Fully abstract models for effectful λ-calculi via category-theoretic logical relations

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preprint & these slides at philipsaville.co.uk

This work

with sum types

A category-theoretic construction that:

takes a [suitable] model of an effectful λ-calculus

. . . and returns an adequate & fully-abstract model

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Adequacy and full abstraction

Contextual equivalence [Morris, Milner,...]

$$\Gamma \vdash M \simeq_{\text{ctx}} M' : \sigma \quad \Longleftrightarrow \quad \mathcal{C}[M] \Downarrow V \iff \mathcal{C}[M'] \Downarrow V$$

$\mathcal{C}[-]$ any closed ground context

Intuition:
swapping M and M'
doesn't affect
observable behaviour

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[*c.f.* soundness and completeness in logic]

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Adequacy: $[\![M]\!] = [\![M']\!] \implies M \simeq_{\mathrm{ctx}} M'$

Full abstraction: $M \simeq_{\mathrm{ctx}} M' \implies [\![M]\!] = [\![M']\!]$

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In an adequate, fully abstract model
semantic equality characterises contextual equivalence

Effectful λ-calculi

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$\rightsquigarrow$ determines a HO language with products & sums

e.g. a HO language with exceptions

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$\rightsquigarrow$ determines an interpretation $[\![\Gamma \vdash M : \sigma]\!] : [\![\Gamma]\!] \to T[\![\sigma]\!]$

This work

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= chosen base types,
effect operations,
& primitives

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semantic model

= CCC with coproducts $\mathcal{M}$
+ strong monad T
+ interpretation
+ conditions on $\mathcal{M}$, interp.

sufficient: full definability
at base types

$\Downarrow$

fully abstract model $\mathrm{OHR}(\mathcal{M})$
of computational λ-calculus + constants + sums

inspired by O'Hearn & Riecke's PCF model, 1995

concrete over $\mathcal{M}$:
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Cranking the handle

signature

e.g. base types $\mathrm{nat}, \mathsf{bool}$
+ primitives $\mathrm{tt}, \mathrm{ff}, \underline{\mathrm{n}}$ for $n \in \mathbb{N}$
+ effect operation $\mathrm{read}, \ldots$

$+$

semantic model

e.g. subcategory Set_κ of Set
+ reader monad R
+ $[\![\mathrm{nat}]\!] = \mathbb{N}$,
$[\![\mathsf{bool}]\!] = \{0, 1\}, \ldots$

not fully abstract!
(Matache & Staton)

$\Downarrow$

fully abstract model
of read-only state

Cranking the handle

signature

e.g. base type real

+ primitive $\underline{f}$ for each measurable f

+ effect operations

$\mathrm{sample}, \mathrm{score}, \mathrm{normalise}, \ldots$

$+$

semantic model

e.g. small sub-CCC of Qbs

+ probability monad

+ $[\![\mathrm{real}]\!] = (\mathbb{R}, \Sigma_{\mathbb{R}})$

$\Downarrow$

fully abstract model
of idealised probabilistic programming language

The OHR construction

Obstruction to full abstraction:

∃ 'bad' morphisms expressing behaviour the syntax cannot

[*c.f.* parallel-or]

The big picture

Obstruction to full abstraction:

$\exists$ 'bad' morphisms expressing behaviour the syntax cannot

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$M \simeq_{\mathrm{ctx}} M' \quad \Longrightarrow \quad [\![M]\!](i) = [\![M']\!](i)$

for all 'program-like' inputs i

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What follows:

1. A general construction for refining models

   [hom-sets and function spaces!]

2. How to instantiate to remove all bad morphisms

A general construction for refining models

Aim: refine a category $\mathcal{M}$ so maps all satisfy certain properties

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Example: category Pred over Set:

objects: pairs $(W \in \mathrm{Set}, \text{predicate } A \subseteq W)$

maps: maps in $\mathcal{M}$ preserving the predicates

Aim: refine a category $\mathcal{M}$ so maps all satisfy certain properties

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The category $\mathrm{Pred}(\mathcal{M})$:

objects: pairs ($W \in \mathcal{M}$, 'relation' on W)

[unary, *n*-ary, varying arity; families of relations,. . .]

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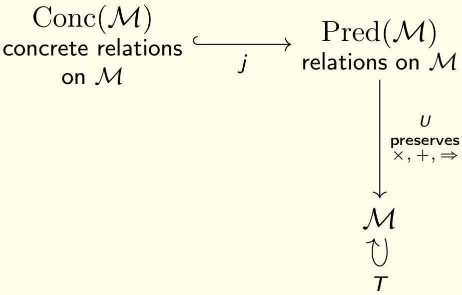

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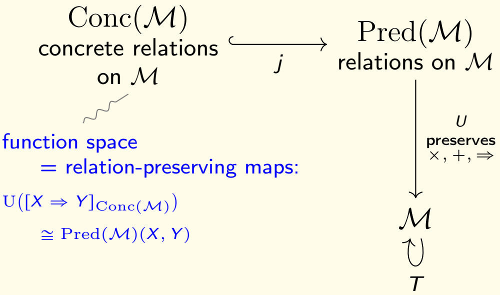

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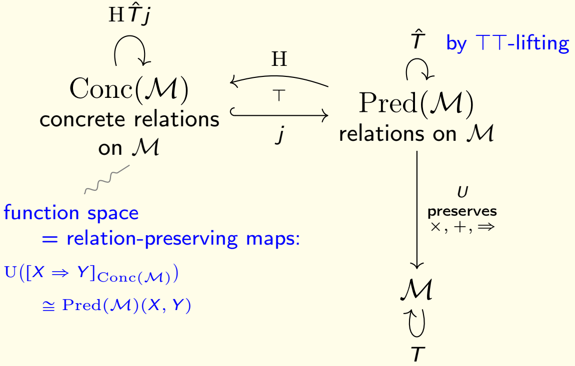

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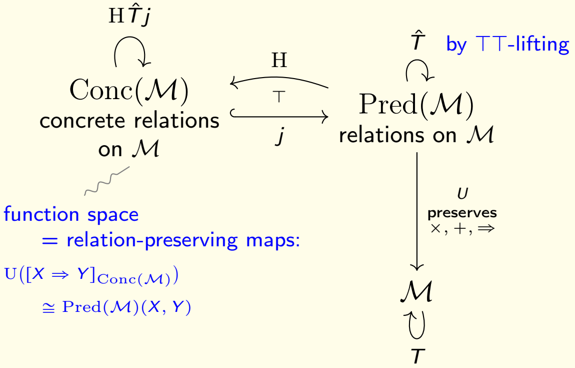

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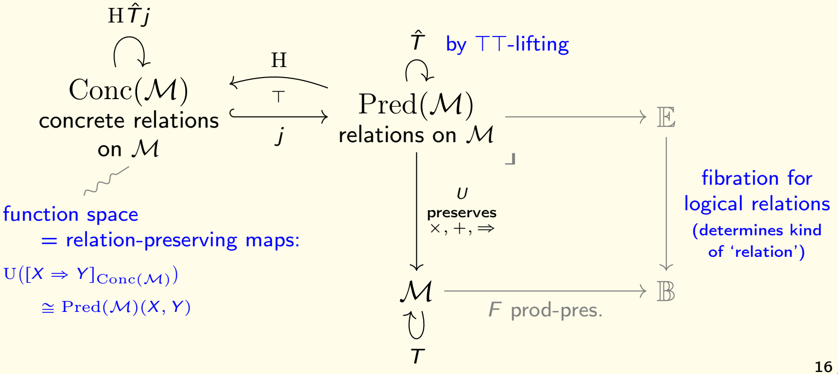
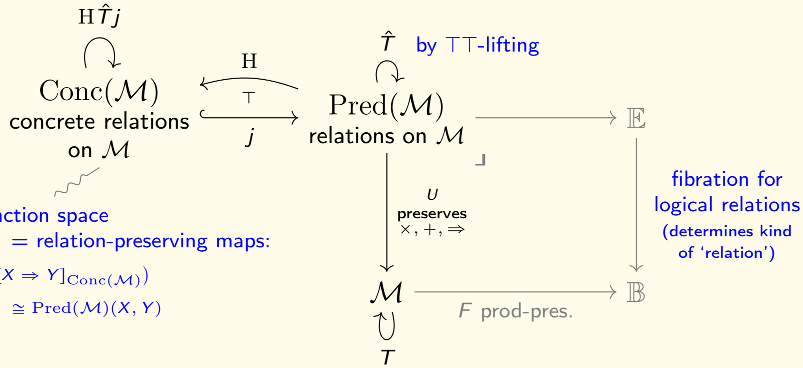


A general construction for refining a model $\mathcal{M}$



function space
= relation-preserving maps:

$\mathrm{U}([X \Rightarrow Y]_{\mathrm{Conc}(\mathcal{M})})$
$\cong \mathrm{Pred}(\mathcal{M})(X, Y)$

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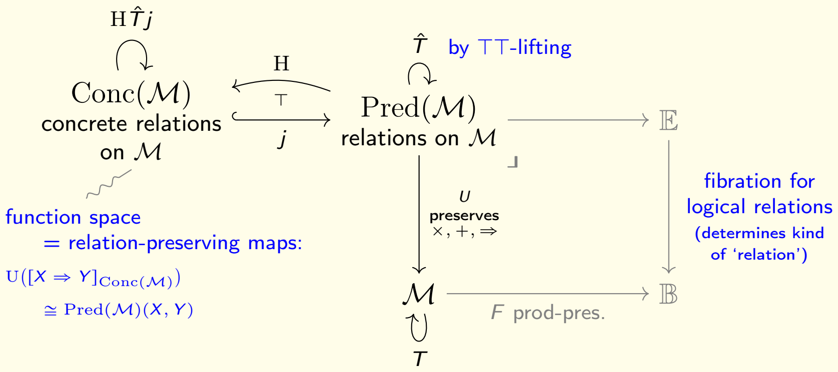

$\rightsquigarrow$ $\mathrm{OHR}(\mathcal{M})$ will be $\mathrm{Conc}(\mathcal{M})$ for a careful choice of "relations"

The big picture

Obstruction to full abstraction:

$\exists$ 'bad' morphisms expressing behaviour the syntax cannot

[*c.f.* parallel-or]

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What follows:

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   [hom-sets and function spaces!]

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Instantiating the general construction

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Lemma: [*c.f.* Curien's "definable separability condition"]

any well-pointed model in which every map $[\![\Gamma]\!] \to T[\![\sigma]\!]$ is definable is fully abstract.

$f = g : X \to Y$
iff
$f \circ \gamma = g \circ \gamma$ for all $\gamma : 1 \to X$

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[Plotkin, Jung & Tiuryn, Alimohamed, ...]

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type-indexed family of relations
compatible with type- & term-formers

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instantiate general construction with a set $\mathbb{I}$ and an interpretation s.t.

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1. objects of $\mathrm{OHR}(\mathcal{M})$ are pairs $\left(W, \{R_i \mid i \in \mathbb{I}\}\right)$ + concreteness

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2. for any logical relation $(L_\sigma \mid \sigma \in \mathrm{Type})$ there exists $i_0 \in \mathbb{I}$ s.t.

$$\left(\begin{array}{c}\text{relation at index } i_0 \\ \text{for interpretation of } \sigma\end{array}\right) = L_\sigma$$

for every type σ

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$$\left(\begin{array}{c} f : [\![\Gamma]\!] \to \mathrm{H}\,\hat{T} j [\![\sigma]\!] \\ \text{in } \mathrm{OHR}(\mathcal{M}) \end{array} \right)$$

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Then:

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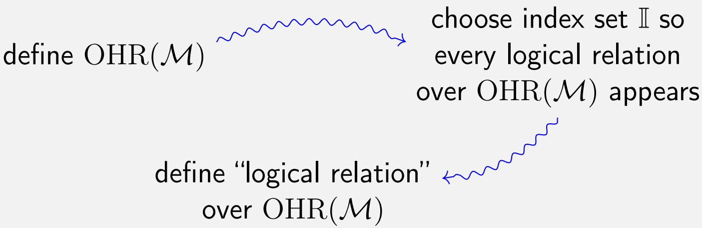

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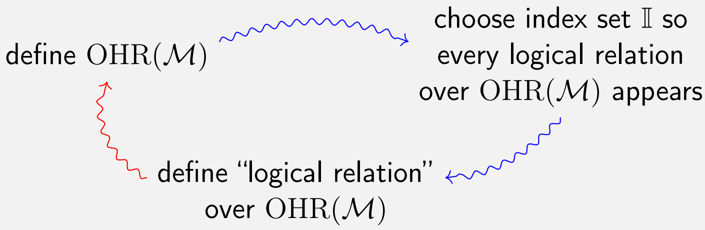

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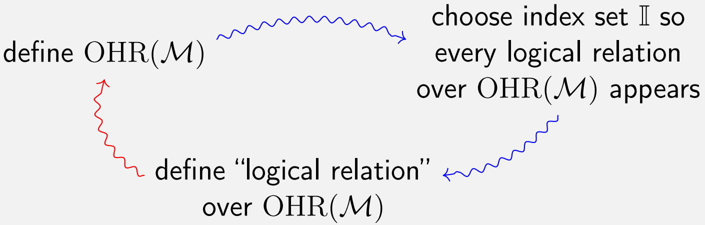

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define $\mathrm{OHR}(\mathcal{M})$ ⇝ choose family $\mathbb{I}$ so every possible relation over $\mathcal{M}$ appears ⇝ identify logical relations over $\mathrm{OHR}(\mathcal{M})$ amongst these relations

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objects: $\big(W \in \mathcal{M}, \{R_i \mid i \in \mathbb{I}\}\big)$ + concreteness | maps: maps in $\mathcal{M}$ preserving all R_i

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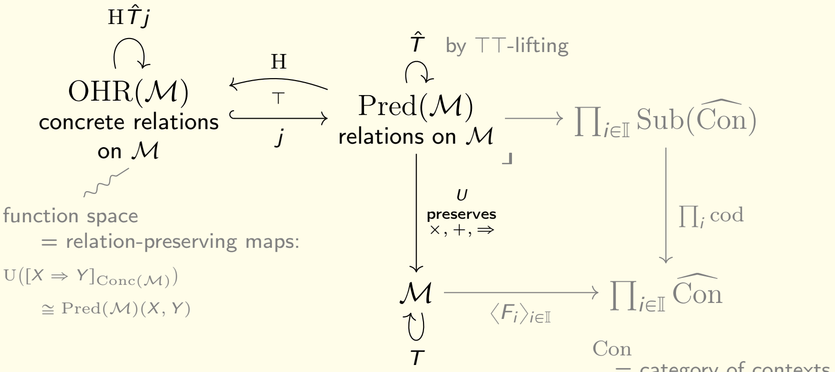
The OHR construction

Choose $\mathbb{I}$ as above, then instantiate general construction as follows:



Codomain fibration on presheaves

$\rightsquigarrow$ relations are Kripke relations of varying arity

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