Category-theoretic syntactic models of programming languages

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Abstract

This dissertation studies the category-theoretic semantics of typed programming languages. It is known that Freyd-categories provide sound and complete semantics for the computational lambda calculus, but a detailed description of the direct model and its corresponding proofs are not present in the literature. The main contribution of this project is the direct formalization of the interpretation of the semantics of the computational lambda calculus in Freyd-categories and the syntactic Freyd-category of the computational lambda calculus and providing detailed proofs of soundness and completeness, and a free property showing that the computational lambda calculus is an internal language of Freyd-categories, as well as the description of a semantically justified translation from the computational lambda calculus to the monadic metalanguage.

Contents

1

Introduction

1.1 Background

Denotational semantics aims to describe the meaning of programming languages by describing programs with mathematical objects. The denotation of a programming language term is built up inductively from the denotation of its subterms, i.e., *compositionally*. The study of such mathematical descriptions can be helpful in understanding programming language concepts in an implementation-independent way, which can be useful in programming language design. It can also be used to formally prove statements about the behaviour of programs, which can be useful in formal program verification.

We can, for example, use a sufficiently faithful mathematical description to prove that two program terms are contextually equivalent, i.e., that we can replace one with another in any program and the observable outcome does not change. Such a statement can be hard to prove syntactically, but with a sound and adequate denotational semantics, it reduces to checking that the corresponding mathematical descriptions agree as in [21]. Studying contextual equivalence can be used in optimizing compiler design to find and justify optimizations that do not change the semantics of the program, for example, as in [7].

One well-known result of denotational semantics is that cartesian closed categories (CCCs) provide a sound and complete semantics for the simply-typed lambda calculus [10]. While such a result is significant from a denotational semantics point of view, it also has category theoretic significance. It shows that the simply-typed lambda calculus provides an *internal language* for CCCs, and we can use it to prove statements about CCCs using the language of the simply-typed lambda calculus [4, Chapter 4].

However, the simply-typed lambda calculus (STLC) is a completely effect-free programming language, which makes it impossible to model certain programming language features, and hard to model others.

The monadic metalanguage (λ_{ml}) is an alternative to the simply-typed lambda calculus, which extends it with *monads*, a general method of adding computational effects, such as printing, reading data or state. In the monadic metalanguage, side-effecting computation has to be explicitly "marked" with monads, similarly to how it would be done in Haskell. The corresponding denotational semantics result is that CCCs with a strong monad provide a sound and complete semantics for the monadic metalanguage [19].

However, in many commonly used programming languages, that is not how side effects are handled: in languages such as OCaml [11], side-effecting computation does not need to be marked explicitly. A different modification of the simply-typed lambda calculus, the computational lambda calculus (λ_{C}), is often used to model that treatment of side effects.

It is known that Freyd-categories provide sound and complete semantics for the computational lambda calculus. The result is sketched in [24], and [13] proves it by a translation of the computational lambda calculus into another language, fine-grain call-by-value, but a direct semantics and a detailed, formal proof of soundness and completeness are not presented in either.

1.2 Outline and contribution of the dissertation

Chapter 2 contains a short review of the category-theoretic interpretations and corresponding syntactic models of the simply-typed lambda calculus in CCCs. Chapter 3 summarizes the corresponding result for the monadic metalanguage and CCCs with a strong monad.

Chapter 4 contains a detailed description of the interpretation and syntactic model of the computational lambda calculus in Freyd categories, with the corresponding proofs of correctness.

Chapter 5 describes and proves how to synthesise a semantically-justified translation from the computational lambda calculus to the monadic metalanguage. The computational lambda calculus can be regarded as a minimal model of Ocaml, and the monadic metalanguage as a minimal model of Haskell, so such a translation can have real-life relevance as it can inform a translation of Ocaml to Haskell.

The main contribution of this dissertation is hence three-fold. Firstly, it is the explicit formalization of the direct syntactic model of λ_C in Freyd-categories. Secondly, it is the formal proof of correctness which is often omitted when similar results are claimed, such as in [13]. Soundness is proved in Theorem 11. Theorem 12 proves that the claimed syntactic closed Freyd-category is indeed a closed Freyd-category, and Theorem 13 uses these results to prove a certain free property. And thirdly, it is the translation and its justification (Theorem 14) from the computational lambda calculus to the monadic metalanguage.

1.3 Related work

The untyped lambda calculus is a Turing-complete model of computation introduced by Church [2]. The simply-typed lambda calculus is a typed, terminating fragment of that, also introduced by Alonzo Church [3]. The simply typed lambda calculus was related to CCCs by Lambek [9]. Until that observation, denotational semantics were largely built on sets and functions.

The monadic metalanguage and its denotational semantics in CCCs with strong monads have been introduced by Moggi [19]. The idea of using monads to organise effects has been a particularly influential one and informed the design of many modern functional programming languages such as Haskell or Agda.

The computational lambda calculus and its denotational semantics in cartesian categories with a strong monad and Kleisli-exponentials are also due to Moggi [18].

Freyd-categories were introduced and related to the computational lambda calculus by Power and Thielecke [23], [24]. A semantics for the fine-grain call-by-value model in Freyd categories and a description of its relation to the computational lambda calculus is presented in [13] by Levy, Power and Thielecke.

2

Simply-typed lambda calculus and cartesian closed categories

This chapter is a short summary of the simply-typed lambda calculus, cartesian closed categories, and the interpretation of the former in the latter. It outlines the key results and concepts and serves as background for understanding the main results from Chapters 4 and 5. Proofs for standard results are omitted, for more detail, see [4].

2.1 Simply-typed lambda calculus

The simply-typed lambda calculus (STLC) is a simple, fully functional programming language that, unlike the untyped lambda calculus, is not Turing complete, and can only describe terminating programs [25]. Nonetheless, it is an important starting point to understanding programming languages and building more complex models, as we will see in the following chapters.

In this dissertation, we work with a version of the STLC parameterized by a **signature**, with a unit type we denote by 1, products, and functions.

Definition 1 (Signature for the STLC)**.** *A* signature $\mathcal{S}$ *for the STLC consists of a set* $\mathcal{S}_{type}$ *of base types, and a set* $\mathcal{S}_{const}$ *describing the constants.* $\mathcal{S}_{const}$ *has elements* (c, τ) *where* c *is the name of the constant and* τ *is a STLC-type.* ▲

Definition 2 (Types of the STLC)**.** *The types of the STLC are given by the following grammar*

$$\tau ::= \beta \mid 1 \mid \tau_1 \times \tau_2 \mid \tau_1 \to \tau_2$$

where $\beta \in \mathcal{S}_{type}$ *ranges over the given base types.* ▲

Definition 3 (Terms of the STLC)**.** *The terms of the STLC are given by the following grammar*

$$E ::= x \mid c \mid () \mid \langle E_1, E_2 \rangle \mid \pi_i E \mid \lambda x.E \mid E_1 E_2$$

where c *ranges over the given constant symbols, i.e.,* $(c, \tau) \in \mathcal{S}_{const}$ *for some* τ*.* ▲

The typing rules are described in Figure 2.1. In the typing rules we use a *context* (often denoted by Γ), which is an ordered list of pairs of variables and types.

The equations of the STLC are defined in Figure 2.2. These describe the semantics of the STLC, and are the minimal congruence generated by the β and η-rules familiar from the untyped lambda calculus. The β-rules describe how the programs reduce, and they correspond to the intuition that the meaning of a program should be preserved if we take one step in the operational semantics. The η-rules describe the extensionality of the STLC.

In what follows, we write terms to represent the α-equivalence class that they belong to, e.g., $x : 1 \vdash \lambda y.yx : (1 \to 1) \to 1$ and $x : 1 \vdash \lambda w.wx : (1 \to 1) \to 1$

$$\frac{}{x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i} \text{(var)}$$

$$\frac{(c, \tau) \in \mathcal{S}_{const}}{\Gamma \vdash c : \tau} \text{(const)}$$

$$\frac{\Gamma \vdash E : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i E : \tau_i} \text{(proj)}$$

$$\frac{\Gamma \vdash E_1 : \tau_1 \quad \Gamma \vdash E_2 : \tau_2}{\Gamma \vdash \langle E_1, E_2 \rangle : \tau_1 \times \tau_2} \text{(pair)}$$

$$\frac{\Gamma, x : \tau_1 \vdash E : \tau_2}{\Gamma \vdash \lambda x.E : \tau_1 \to \tau_2} \text{(abst)}$$

$$\frac{\Gamma \vdash E_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash E_2 : \tau_1}{\Gamma \vdash E_1 E_2 : \tau_2} \text{(app)}$$

Figure 2.1: Simply-typed lambda calculus over a signature

represent the same terms. We use $E_1[x \mapsto E_2]$ to denote the capture-avoiding substitution of the free variable x in the term E_1 with the term E_2. As these are standard practices for working with the lambda calculus, these are not detailed here, for more detail see [14].

2.2 Cartesian closed categories

Definition 4 (Terminal object). *In a category $\mathcal{C}$, an object* 1 *is a* terminal object *iff for every object X there is a unique morphism $! : X \to 1$.*

$$X \overset{!}{\dashrightarrow} 1$$

▲

Definition 5 (Binary product). *In a category $\mathcal{C}$, given two objects A_1, A_2, the* binary product *of A_1 and A_2 (if it exists) is an object $A_1 \times A_2$, and two morphisms:*

$$\frac{\Gamma \vdash E_1 : \tau_1 \quad \Gamma \vdash E_2 : \tau_2 \quad i \in \{1,2\}}{\Gamma \vdash \pi_i \langle E_1, E_2 \rangle \equiv E_i : \tau_i} \qquad (\text{prod}\beta)$$

$$\frac{\Gamma, x : \tau_1 \vdash E_1 : \tau_2 \quad \Gamma \vdash E_2 : \tau_1}{\Gamma \vdash (\lambda x.E_1)E_2 \equiv E_1[x \mapsto E_2] : \tau_2} \qquad (\text{fn}\beta)$$

$$\frac{\Gamma \vdash E : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 E, \pi_2 E \rangle \equiv E : \tau_1 \times \tau_2} \qquad (\text{prod}\eta)$$

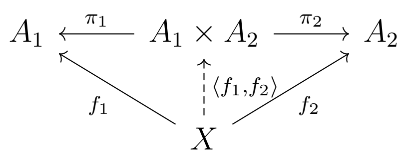
$$\frac{\Gamma \vdash E : \tau_1 \to \tau_2 \quad x \text{ is fresh in } E}{\Gamma \vdash \lambda x.Ex \equiv E : \tau_1 \to \tau_2} \qquad (\text{fn}\eta)$$

$$\frac{\Gamma \vdash E : 1}{\Gamma \vdash () \equiv E : 1} \qquad (\text{unit})$$

Together with the equivalence relation rules (reflexivity, symmetry, transitivity), and congruence rules for each constructor.

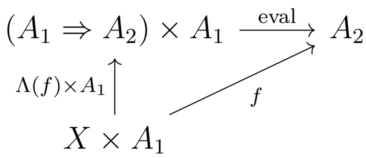
Figure 2.2: Equations of the STLC

$\pi_1 : (A_1 \times A_2) \to A_1$ and $\pi_2 : (A_1 \times A_2) \to A_2$, such that for any object X, and any morphisms $f_1 : X \to A_1$, $f_2 : X \to A_2$, there is a unique morphism $\langle f_1, f_2 \rangle : X \to (A_1 \times A_2)$ such that $\pi_1 \circ \langle f_1, f_2 \rangle = f_1$ and $\pi_2 \circ \langle f_1, f_2 \rangle = f_2$.



▲

Definition 6 (Exponential)**.** *In a category $\mathcal{C}$ in which all binary products exist, given two objects A_1, A_2 the* exponential *(if it exists) is an object $A_1 \Rightarrow A_2$ together with a morphism* $\text{eval} : ((A_1 \Rightarrow A_2) \times A_1) \to A_2$ *such that for any object X and morphism $f : X \times A_1 \to A_2$, there is a unique map $\Lambda(f) : X \to (A_1 \Rightarrow A_2)$ such* *that the following diagram commutes.*



▲

Definition 7 (Cartesian closed category)**.** *A* cartesian closed category *(CCC) is a category with a terminal object, where all binary products, and all exponentials exist.* ▲

Note that a cartesian closed category might have multiple possible choices for products and exponentials, in what follows, when referring to a particular CCC, we assume a particular choice of products and exponentials.

Example 1. *The category* **Set** *where the objects are sets and the morphisms are functions between sets is a cartesian closed category. The terminal object is the one-element set. Given two objects corresponding to sets X_1 and X_2, their binary product object is given by the Cartesian product $X_1 \times X_2$, and their exponential object is given by the set of all functions from X_1 to X_2.* ▲

2.3 Connection

An interpretation is a mapping from the types and terms of a programming language to mathematical objects. In our case, the types will be mapped to objects in a category, and the typed terms will be mapped to morphisms.

An interpretation $[\![-]\!]$ of a typed language $\mathcal{L}$ is *sound* with respect to an equational theory $\Gamma \vdash - \equiv - : \tau$, iff

$$(\Gamma \vdash M_1 \equiv M_2 : \tau) \Rightarrow ([\![\Gamma \vdash M_1 : \tau]\!] = [\![\Gamma \vdash M_2 : \tau]\!]).$$

An interpretation $[\![-]\!]$ of a typed language $\mathcal{L}$ is *complete* with respect to an equational theory $\Gamma \vdash - \equiv - : \tau$, iff

$$([\![\Gamma \vdash M_1 : \tau]\!] = [\![\Gamma \vdash M_2 : \tau]\!]) \Rightarrow (\Gamma \vdash M_1 \equiv M_2 : \tau).$$

Subsection 2.3.1 describes an interpretation of the STLC that is sound with respect to the equational theory from Figure 2.2. Subsection 2.3.2 then describes the *syntactic* CCC of the STLC, i.e., a CCC that is “built from” the syntax of the STLC and the equations, and in which interpretation of the STLC is complete.

The existence of this CCC proves the following completeness result of interpretations of the STLC in CCCs: if the interpretation of two terms agrees in all CCCs, they are equal with respect to the equational theory. Together with the soundness result, this proves that two terms are equal with respect to the equational theory iff their interpretations agree in all CCCs.

Subsection 2.3.3 formalizes the free property the interpretation of the STLC in the syntactic CCC has.

2.3.1 Interpretation of the STLC in CCCs

Definition 8 (Interpretation of a signature in a CCC)**.** *Given a signature $\mathcal{S} = (\mathcal{S}_{type}, \mathcal{S}_{const})$, and a CCC $\mathcal{C}$ with chosen products and exponentials, an* interpretation *of $\mathcal{S}$ in $\mathcal{C}$ is a map $i_{type} : \mathcal{S}_{type} \to ob(\mathcal{C})$ extended to a mapping of all types to objects as in Figure 2.3, and a map i_{const} that maps a constant $(c, \tau) \in \mathcal{S}_{const}$ to a morphism $1 \to [\![\tau]\!]$ in $\mathcal{C}$, that is extended to a mapping from all terms of the STLC with that signature, as in Figure 2.3.* ▲

Note that this interpretation maps types to objects and terms $\Gamma \vdash E : \tau$ with context $\Gamma = [x_1 : \tau_1, x_2 : \tau_2, \ldots x_n : \tau_n]$ to morphisms from $[\![((\tau_1 \times \tau_2) \times \ldots) \times \tau_n]\!]$ to $[\![\tau]\!]$.

$$
\begin{aligned}
[\![\beta]\!] &= i_{type}(\beta) \\
[\![1]\!] &= 1 \\
[\![\tau_1 \times \tau_2]\!] &= [\![\tau_1]\!] \times [\![\tau_2]\!] \\
[\![\tau_1 \to \tau_2]\!] &= [\![\tau_1]\!] \Rightarrow [\![\tau_2]\!] \\
\\
[\![\lozenge]\!] &= 1 \\
[\![x_1 : \tau_1, \dots x_n : \tau_n]\!] &= (([\![\tau_1]\!] \times [\![\tau_2]\!]) \times \dots) \times [\![\tau_n]\!] \\
\\
[\![\Gamma \vdash c : \tau]\!] &= i_{const}(c, \tau) \circ ! \\
[\![\Gamma \vdash x_i : \tau_i]\!] &= \left([\![\Gamma]\!] \xrightarrow{\pi_i} [\![\tau_i]\!]\right) \\
[\![\Gamma \vdash () : 1]\!] &= \left([\![\Gamma]\!] \xrightarrow{!} [\![\tau_i]\!]\right) \\
[\![\Gamma \vdash \lambda x.E : \tau_1 \to \tau_2]\!] &= \Lambda\left([\![\Gamma]\!] \times [\![\tau_1]\!] \xrightarrow{[\![\Gamma, x:\tau_1 \vdash E:\tau_2]\!]} [\![\tau_2]\!]\right) \\
[\![\Gamma \vdash \langle E_1, E_2 \rangle : \tau_1 \times \tau_2]\!] &= \left([\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash E_1:\tau_1]\!], [\![\Gamma \vdash E_2:\tau_2]\!] \rangle} [\![\Gamma]\!]\right) \\
[\![\Gamma \vdash \pi_i E : \tau_i]\!] &= \left([\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash E:\tau_1 \times \tau_2]\!]} [\![\tau_1]\!] \times [\![\tau_2]\!] \xrightarrow{\pi_i} [\![\tau_i]\!]\right) \\
[\![\Gamma \vdash E_1 E_2 : \tau_2]\!] &= \left([\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash E_1:\tau_1 \to \tau_2]\!], [\![\Gamma \vdash E_2:\tau_2]\!] \rangle} ([\![\tau_1]\!] \Rightarrow [\![\tau_2]\!]) \times [\![\tau_1]\!] \xrightarrow{\text{eval}} [\![\tau_2]\!]\right)
\end{aligned}
$$

Figure 2.3: Interpretation of the STLC in a CCC

Theorem 1. *The interpretation of the STLC in any CCC, described in Definition 8, is sound with respect to the equational theory described in Figure 2.2.* □

2.3.2 Syntactic CCC of the STLC

Given a signature $\mathcal{S}$, the syntactic CCC of the STLC with that signature is the category $\mathcal{F}[\mathcal{S}]$ with:

- **Objects** given by types of the STLC.

- **Morphisms** between objects τ_1 and τ_2 given by equivalence classes of well-typed terms E of the STLC with a fixed variable x,

$$x : \tau_1 \vdash E : \tau_2,$$

quotiented by α-renaming and the equational theory described in Figure 2.2.

Note that for clarity, we abuse notation by using different variable names $x, x_1, x_2, \ldots, y \ldots$ for naming the one fixed variable.

- **Identity morphism** of an object τ is given by $x : \tau \vdash x : \tau$.
- **Composition** is given by substitution:

$$(y : \tau_2 \vdash E_2 : \tau_3) \circ (x : \tau_1 \vdash E_1 : \tau_2) = (x : \tau_1 \vdash E_2[y \mapsto E_1] : \tau_3).$$

Theorem 2. *$\mathcal{F}[\mathcal{S}]$ is a CCC.* □

There is a natural interpretation ι of the STLC in $\mathcal{F}[\mathcal{S}]$ with

$$\iota(\beta) = \beta \qquad \text{for } \beta \text{ in } \mathcal{S}_{type}$$
$$\iota(c) = (\vdash c : \tau) \qquad \text{for a constant } c \text{ of type } \tau \text{ in } \mathcal{S}_{const}.$$

extended to all types and terms as in Definition 8.

Theorem 3. *The interpretation ι of the STLC in $\mathcal{F}[\mathcal{S}]$ is complete with respect to the equational theory described in Figure 2.2.*

Proof. This statement holds by the definition of the category, as two terms $x : \tau_1 \vdash E_1 : \tau_2$ and $x : \tau_1 \vdash E_2 : \tau_2$ are in the same equivalence class iff $x : \tau_1 \vdash E_1 \equiv E_2 : \tau_2$. □

For more on the STLC and CCCs, see Chapter 4 of Categories for Types [5]. For a full proof of the soundness and completeness stated in this chapter, see [6].

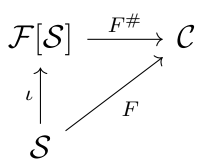
2.3.3 Free property

Definition 9 (Strict cartesian closed functor). *Given two cartesian closed categories $\mathcal{C}_1, \mathcal{C}_2$ with chosen products and exponentials, a strict CC-functor $\mathcal{C}_1 \to \mathcal{C}_2$ is a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ that strictly preserves cartesian closed structure, i.e., for any objects τ_1, τ_2, τ, and morphisms f_1, f_2, f, and for $i \in \{1, 2\}$,*

$$
\begin{aligned}
F(1_{\mathcal{C}_1}) &= 1_{\mathcal{C}_2} \\
F(\tau_1 \times_{\mathcal{C}_1} \tau_2) &= F(\tau_1) \times_{\mathcal{C}_2} F(\tau_2) \\
F(\tau_1 \Rightarrow_{\mathcal{C}_1} \tau_2) &= F(\tau_1) \Rightarrow_{\mathcal{C}_2} F(\tau_2) \\
F(!^{\tau}_{\mathcal{C}_1}) &= !^{F(\tau)}_{\mathcal{C}_2} \\
F(\pi^{\tau}_{\mathcal{C}_1,i}) &= \pi^{F(\tau)}_{\mathcal{C}_2,i} \\
F(\langle f_1, f_2 \rangle_{\mathcal{C}_1}) &= \langle F(f_1), F(f_2) \rangle_{\mathcal{C}_2} \\
F(\mathrm{eval}^{\tau}_{\mathcal{C}_1}) &= \mathrm{eval}^{F(\tau)}_{\mathcal{C}_2} \\
F(\Lambda_{\mathcal{C}_1}(f)) &= \Lambda_{\mathcal{C}_2}(F(f)).
\end{aligned}
$$

▲

Definition 10 (Free CCC over a signature). *Given a signature $\mathcal{S} = (\mathcal{S}_{type}, \mathcal{S}_{const})$ a CCC $\mathcal{F}[\mathcal{S}]$ is* free *over $\mathcal{S}$ iff there exists an interpretation ι of $\mathcal{S}$ in $\mathcal{F}[\mathcal{S}]$ such that for any CCC $\mathcal{C}$, and any interpretation F of $\mathcal{S}$ in $\mathcal{C}$, there is a unique strict CC-functor $F^{\#}$ such that the following diagram commutes:*



i.e., for any $\beta \in \mathcal{S}_{type}$*,* $F^{\#}(\iota(\beta)) = F(\beta)$ *and any* $c \in \mathcal{S}_{const}$*,* $F^{\#}(\iota(c)) = F(c)$*.* ▲

To motivate calling it free, consider the following adjunction.

Definition 11 (Restricted signature for the STLC)**.** *A* restricted signature for the STLC $\mathcal{S}$ *consists of a set* $\mathcal{S}_{type}$ *of base types, and a set* $\mathcal{S}_{const}$ *describing the constants.* $\mathcal{S}_{const}$ *has elements* (c, τ) *where* c *is the name of the constant and* $\tau \in \mathcal{S}_{type}$*.* ▲

Note that unlike before, constants cannot be of arbitrary types, only of base types.

Definition 12 ($\mathbf{Sig}^{-}$)**.** *Let* $\mathbf{Sig}^{-}$ *be the category of restricted signatures: the objects are restricted signatures of the STLC, and a morphism from* $(\mathcal{S}_{type,1}, \mathcal{S}_{const,1})$ *to* $(\mathcal{S}_{type,2}, \mathcal{S}_{const,2})$ *is a mapping of base types* $F_{types} : \mathcal{S}_{type,1} \to \mathcal{S}_{type,2}$ *and a mapping of constants* $F_{const} : \mathcal{S}_{const,1} \to \mathcal{S}_{const,2}$ *respecting the types.* ▲

Definition 13 (**CCC**)**.** *Let* **CCC** *be the category of CCCs: the objects are small CCCs with chosen products and exponentials and the morphisms are strict cartesian closed functors.* ▲

Given $\mathcal{C} \in \mathbf{CCC}$, let $U\mathcal{C} \in \mathbf{Sig}$ be the underlying signature of $\mathcal{C}$, in particular, let $U\mathcal{C} = (\mathcal{S}_{types}, \mathcal{S}_{const})$ where $\mathcal{S}_{types} = \text{ob}(\mathcal{C})$ and $\mathcal{S}_{const} = \bigcup_{X \in \text{ob}(\mathcal{C})} \bigcup_{f \in \mathcal{C}(1,X)} \{(f, X)\}$, i.e., we choose all types as base types and all morphisms from the terminal object to a base type as constants.

For this special case of restricted signatures, we can rephrase the uniqueness requirement in Definition 10 as follows: for any $\mathcal{S} \in \mathbf{Sig}^{-}$,$(\mathcal{F}[\mathcal{S}], \iota : \mathcal{S} \to UF^{\#}\mathcal{S})$ is such that $\mathcal{F}[\mathcal{S}] \in \mathbf{CCC}$ and for any $\mathcal{C} \in \mathbf{CCC}$ and functor $F : \mathcal{S} \to U\mathcal{C}$, there is a unique $F^{\#} : \mathcal{F}[\mathcal{S}] \to \mathcal{C}$ in **CCC** such that the following diagram commutes in $\mathbf{Sig}^{-}$:

$$\begin{array}{ccc} U\mathcal{F}[\mathcal{S}] & \overset{UF^{\#}}{\dashrightarrow} & U\mathcal{C} \\ {\scriptstyle\iota}\uparrow & \nearrow_{F} & \\ \mathcal{S} & & \end{array} .$$

This is exactly the universality condition of the following adjunction.

$$\mathbf{Sig}^- \underset{\mathcal{F}[-]}{\overset{U}{\rightleftarrows}} \mathbf{CCC} \qquad (\top)$$

Our construction is more general as it allows for constants of any type, but this restricted case illustrates the reason for naming it free.

Theorem 4. $\mathcal{F}[\mathcal{S}]$ *with* ι *is the free CCC over* $\mathcal{S}$.

Proof sketch. $\mathcal{F}[\mathcal{S}]$ is a CCC by Theorem 2.

$F^{\#}$ is a mapping from objects and morphisms of $\mathcal{F}[\mathcal{S}]$ so from types and equivalence classes of typed lambda terms (with a one-variable context) to objects of $\mathcal{C}$.

The requirement that the above diagram commutes enforces the behaviour of $F^{\#}$ on base types. The requirement that $F^{\#}$ is a CC-functor extends the behaviour to the terminal object, products and exponentials. Similarly, the requirement that the above diagram commutes enforces the behaviour of $F^{\#}$ on constants, and we can inductively extend this to all morphisms using the requirement that $F^{\#}$ is a CC-functor. Hence if such an $F^{\#}$ exists, it is unique, and it maps on object corresponding to the type τ to $[\![\tau]\!]$ and a morphism corresponding to $x : \tau_1 \vdash E : \tau_2$ to $[\![x : \tau_1 \vdash E : \tau_2]\!]$ for the interpretation of the STLC with that signature as defined in Definition 8.

Now Theorem 1 can be used to see that this gives a well-defined functor on morphisms, i.e., that if two terms correspond to the same morphism in $\mathcal{F}[\mathcal{S}]$, i.e., if they are equivalent in the congruence from above, then they are mapped to the same object by $\mathcal{F}[\mathcal{S}]$. Finally, it remains to check that $F^{\#}$ is indeed a strict CC-functor: preservation of composition can be proved by induction on the second morphism of the composition and preservation of identities, strict preservation of the terminal object and binary products all follow directly from the definition of $F^{\#}$. $\square$

3

Monadic metalanguage and cartesian closed categories with a strong monad

This chapter is a short summary of the monadic metalanguage, strong monads, and the interpretation of the monadic metalanguage in CCCs with a strong monad.

3.1 Monadic metalanguage

The monadic metalanguage (λ_{ml}) was introduced by Moggi [19] as a minimalist programming language that allows for the modelling of side-effecting computation. It extends the STLC by adding a new type constructor T that describes monadic computation. Intuitively, a computation of type $T\tau$ means a computation of type τ that potentially has side effects of kind T.

This treatment of side effects is similar to that of Haskell, where a print function has type `putStrLn :: String -> IO ()`. Here `IO` has the same role as the monad T in our metalanguage.

To combine monadic computations, we can use the let-binding in our language: let $x \Leftarrow E_1$ in E_2 corresponds to the intuition of "perform the computation of E_1 together with all its side effects, bind the resulting value of E_1 and perform the computation E_2 with the resulting value substituted for x". The corresponding operator in Haskell is `>>=`, for example, we could combine two printing operations as follows:

```
putStr "hello " >>= (\ x  -> putStr "world")
```

(Note that here x will be bound to the result of the first `putStr` statement, which has type `()` and we do not use it in the second `putStr` statement.)

In the monadic metalanguage, to create a monadic term of type $T\tau$ from a term of type τ without adding any actual side effect, we can use the $[-]_T$ construct. This corresponds to `return` in Haskell.

```
(return ()) >>= (\x -> putStr "hello world")
```

A signature for the monadic metalanguage is similar to that of the simply-typed lambda calculus: a set of base types and a set of constants, but now these might have a monadic type. For example, a possible signature could be $\mathcal{S}_{type} = \{\texttt{bool}, \texttt{string}\}$, $\mathcal{S}_{const} = \{(\texttt{true}, \texttt{bool}), (\texttt{false}, \texttt{bool}), (\texttt{print}, \texttt{string} \to T1)\}$.

Definition 14 (Signature for λ_{ml})**.** *A signature for* λ_{ml} $\mathcal{S}$ *consists of a set* $\mathcal{S}_{type}$ *of base types, and a set* $\mathcal{S}_{const}$ *describing the constants.* $\mathcal{S}_{const}$ *has elements* (c, τ) *where* c *is the name of the constant and* τ *is a monadic metalanguage type.* ▲

Definition 15 (Types of λ_{ml})**.** *The types of the monadic metalanguage are given by the following grammar*

$$\tau ::= \beta \mid 1 \mid T\tau \mid \tau_1 \times \tau_2 \mid \tau_1 \to \tau_2$$

$$\frac{}{x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i} \text{ (var)}$$

$$\frac{(c, \tau) \in \mathcal{S}_{const}}{\Gamma \vdash c : \tau} \text{ (const)}$$

$$\frac{\Gamma \vdash E : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i E : \tau_i} \text{ (proj)}$$

$$\frac{\Gamma \vdash E_1 : \tau_1 \quad \Gamma \vdash E_2 : \tau_2}{\Gamma \vdash \langle E_1, E_2 \rangle : \tau_1 \times \tau_2} \text{ (pair)}$$

$$\frac{\Gamma \vdash E : T}{\Gamma \vdash [E]_T : T\tau} \text{ (return)}$$

$$\frac{\Gamma \vdash E_1 : T\tau_1 \quad \Gamma, x : \tau_1 \vdash E_2 : T\tau_2}{\Gamma \vdash \text{let } x \Leftarrow E_1 \text{ in } E_2 : T\tau_2} \text{ (let)}$$

$$\frac{\Gamma, x : \tau_1 \vdash E : \tau_2}{\Gamma \vdash \lambda x.E : \tau_1 \to \tau_2} \text{ (abst)}$$

$$\frac{\Gamma \vdash E_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash E_2 : \tau_1}{\Gamma \vdash E_1 E_2 : \tau_2} \text{ (app)}$$

Figure 3.1: Monadic metalanguage

where $\beta \in \mathcal{S}_{type}$ *ranges over the given base types.* ▲

Definition 16 (Terms of the monadic metalanguage)**.** *The terms of the monadic metalanguage are given by the following grammar*

$$E ::= x \mid c \mid () \mid \langle E_1, E_2 \rangle \mid \pi_i E \mid \lambda x.E \mid E_1 E_2 \mid [E]_T \mid let\ x \Leftarrow E_1\ in\ E_2$$

where c *ranges over the given constant symbols, i.e.,* $(c, \tau) \in \mathcal{S}_{const}$ *for some* τ*.* ▲

The typing rules are described in Figure 3.1.

The equations of λ_{ml} are defined in Figure 3.2. They differ from the rules for the STLC by the addition of the (letβ), (letη) and the (assoc) rules.

$$\frac{\Gamma \vdash E_1 : \tau_1 \quad \Gamma \vdash E_2 : \tau_2 \quad i \in \{1,2\}}{\Gamma \vdash \pi_i\langle E_1, E_2\rangle \equiv E_i : \tau_i} \quad (\text{prod}\beta)$$

$$\frac{\Gamma, x : \tau_1 \vdash E_1 : \tau_2 \quad \Gamma \vdash E_2 : \tau_1}{\Gamma \vdash (\lambda x.E_1)E_2 \equiv E_1[x \mapsto E_2] : \tau_2} \quad (\text{fn}\beta)$$

$$\frac{\Gamma \vdash E_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash E_2 : \tau_2}{\Gamma \vdash \text{let } x \Leftarrow [E_1]_T \text{ in } E_2 \equiv E_2[x \mapsto E_1] : \tau_2} \quad (\text{let}\beta)$$

$$\frac{\Gamma \vdash E : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 E, \pi_2 E\rangle \equiv E : \tau_1 \times \tau_2} \quad (\text{prod}\eta)$$

$$\frac{\Gamma \vdash E : \tau_1 \to \tau_2 \quad x \text{ not free in } E}{\Gamma \vdash \lambda x.Ex \equiv E : \tau_1 \to \tau_2} \quad (\text{fn}\eta)$$

$$\frac{\Gamma \vdash E : T\tau}{\Gamma \vdash \text{let } x \Leftarrow E \text{ in } [x]_T \equiv E : T\tau} \quad (\text{let}\eta)$$

$$\frac{\Gamma \vdash E : 1}{\Gamma \vdash () \equiv E : 1} \quad (\text{unit})$$

$$\frac{\Gamma \vdash E_1 : T\tau_1 \qquad \Gamma, x : \tau_1 \vdash E_2 : T\tau_2 \Gamma, y : \tau_2 \vdash E_3 : T\tau_3}{\begin{array}{l}\Gamma \vdash \text{let } y \Leftarrow (\text{let } x \Leftarrow E_1 \text{ in } E_2) \text{ in } E_3 \equiv \\ \qquad \text{let } x \Leftarrow E_1 \text{ in } (\text{let } y \Leftarrow E_2 \text{ in } E_3) : T\tau_3\end{array}} \quad (\text{assoc})$$

Together with the equivalence relation rules (reflexivity, symmetry, transitivity), and congruence rules for each constructor.

Figure 3.2: Equations of λ_{ml}

3.2 Strong monads

The previous section introduced monads as a programming language concept. This section presents monads in category theory. It presents them in Kleisli form, which is equivalent to their standard definition [16] but, as we will see later, aligns more directly with monads in the monadic metalanguage.

Definition 17 (Monad in Kleisli form)**.** *Given a category* $\mathcal{C}$*, a* monad in Klesli form *is a triple* $(T, \eta, (\cdot)^\dagger)$*, where:*

- $T : ob(\mathcal{C}) \to ob(\mathcal{C})$

- *for each object* X, $\eta_X \in \mathcal{C}(X, TX)$
- *for all pairs of objects* X, Y, $(\cdot)^{\dagger}_{X,Y} : \mathcal{C}(X, TY) \to \mathcal{C}(TX, TY)$

satisfying the following axioms:

- $\eta_X^{\dagger} = \mathrm{id}_X$
- $f^{\dagger} \circ \eta_X = f$
- $g^{\dagger} \circ f^{\dagger} = (g^{\dagger} \circ f)^{\dagger}$. ▲

The following theorem illustrates the importance of monads from a category-theoretic point of view.

Theorem 5. *[1] Every adjunction* $L \dashv R$ *gives rise to a monad* $R \circ L$. □

Definition 18 (Strength of a monad)**.** *Given a monad* $(T, \eta, (\cdot)^{\dagger})$, *a* strength *for* T *is a natural transformation with components*

$$\mathrm{st}_{A,B} : A \times TB \to T(A \times B)$$

satisfying the strength axioms from [8]. ▲

Definition 19 (Strong monad)**.** *A* strong monad *is a monad with a strength.* ▲

As we will see below, strong monads can be used to describe the semantics of the monadic metalanguage. In particular, η corresponds to *returning* a value, i.e., making a monadic term from a term without adding any side effects, and $(\cdot)^{\dagger}$ corresponds to sequentially composing computations. The strength of a monad then describes how to combine a term and a monadic term into a single monadic term. As an illustration of the strength of a monad, consider the following theorem.

Theorem 6. *[17] Every monad in* **Set** *with the Cartesian product has a unique strength given by*

$$\mathrm{st}_{X,Y}(x, y_m) = T(\lambda y.\langle x, y\rangle)(y_m).$$

□

3.3 Connection

This section describes how to extend the results for the STLC and CCCs to the case of the monadic metalanguage. The results are due to Moggi [19].

3.3.1 Interpretation of the monadic metalanguage in CCCs with a strong monad

Definition 20 (Interpretation of a signature in a CCC with a strong monad)**.** *Given a signature $\mathcal{S} = (\mathcal{S}_{type}, \mathcal{S}_{const})$, and a CCC $\mathcal{C}$ with chosen products and exponentials and a strong monad $(T, \eta, (\cdot)^\dagger, \mathrm{st})$, an* interpretation *of $\mathcal{S}$ in $\mathcal{C}$ is a map $i_{type} : \mathcal{S}_{type} \to ob(\mathcal{C})$ extended to a mapping of all types to objects as in Figure 3.3, and a map i_{const} that maps a constant $(c, \tau) \in \mathcal{S}_{const}$ to a morphism $1 \to [\![\tau]\!]$ in $\mathcal{C}$, that is extended to a mapping from all terms of λ_{ml} with that signature, as in Figure 3.3.* ▲

Theorem 7. *[19] The interpretation of λ_{ml} in any CCC with a strong monad, described in Definition 20 is sound with respect to the equational theory described in Figure 3.2.* □

$$
\begin{aligned}
[\![\beta]\!] &= i_{type}(\beta) \\
[\![1]\!] &= 1 \\
[\![\tau_1 \times \tau_2]\!] &= [\![\tau_1]\!] \times [\![\tau_2]\!] \\
[\![\tau_1 \to \tau_2]\!] &= [\![\tau_1]\!] \Rightarrow [\![\tau_2]\!] \\
[\![T\tau]\!] &= T[\![\tau]\!]
\end{aligned}
$$

$$
\begin{aligned}
[\![\Diamond]\!] &= 1 \\
[\![x_1 : \tau_1, \dots x_n : \tau_n]\!] &= (([\![\tau_1]\!] \times [\![\tau_2]\!]) \times \dots) \times [\![\tau_n]\!]
\end{aligned}
$$

$$
\begin{aligned}
[\![\Gamma \vdash c : \tau]\!] &= i_{const}(c, \tau) \circ ! \\
[\![\Gamma \vdash x_i : \tau_i]\!] &= \left([\![\Gamma]\!] \xrightarrow{\pi_i} [\![\tau_i]\!]\right) \\
[\![\Gamma \vdash () : 1]\!] &= \left([\![\Gamma]\!] \xrightarrow{!} [\![\tau_i]\!]\right) \\
[\![\Gamma \vdash \lambda x.E : \tau_1 \to \tau_2]\!] &= \Lambda\left([\![\Gamma]\!] \times [\![\tau_1]\!] \xrightarrow{[\![\Gamma, x:\tau_1 \vdash E:\tau_2]\!]} [\![\tau_2]\!]\right) \\
[\![\Gamma \vdash \langle E_1, E_2\rangle : \tau_1 \times \tau_2]\!] &= \left([\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash E_1:\tau_1]\!], [\![\Gamma \vdash E_2:\tau_2]\!]\rangle} [\![\Gamma]\!]\right) \\
[\![\Gamma \vdash \pi_i E : \tau_i]\!] &= \left([\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash E:\tau_1 \times \tau_2]\!]} [\![\tau_1]\!] \times [\![\tau_2]\!] \xrightarrow{\pi_i} [\![\tau_i]\!]\right) \\
[\![\Gamma \vdash E_1 E_2 : \tau_2]\!] &= \\
&\left([\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash E_1:\tau_1 \to \tau_2]\!], [\![\Gamma \vdash E_2:\tau_1]\!]\rangle} ([\![\tau_1]\!] \Rightarrow [\![\tau_2]\!]) \times [\![\tau_1]\!] \xrightarrow{\text{eval}} [\![\tau_2]\!]\right) \\
[\![\Gamma \vdash [E]_T : T\tau]\!] &= \eta_\tau \circ [\![\Gamma \vdash E : \tau]\!] \\
[\![\Gamma \vdash \text{let } x \Leftarrow E_1 \text{ in } E_2 : T\tau_2]\!] &= [\![\Gamma, x : \tau_1 \vdash E_2 : T\tau_2]\!]^\dagger \circ \text{st}_{\tau_1,\tau_2} \circ \langle \text{id}_{\tau_1}, [\![\Gamma \vdash E_1 : \tau_1]\!]\rangle
\end{aligned}
$$

Figure 3.3: Interpretation of the monadic metalanguage in a CCC with a strong monad

3.3.2 Syntactic CCC with strong monad of the Monadic Metalanguage

Given a signature $\mathcal{S}$, the syntactic CCC with a strong monad of λ_{ml} with that signature is the category $\mathcal{F}[\mathcal{S}]$ with:

- **Objects** given by the types of the λ_{ml}.
- **Morphisms** between objects τ_1 and τ_2 given by equivalence classes of well-

typed terms E of λ_{ml} with

$$x : \tau_1 \vdash E : \tau_2,$$

quotiented by the equational theory above.

- **Identity morphism** of an object τ is given by $x : \tau \vdash x : \tau$.
- **Composition** is given by substitution:

$$(y : \tau_2 \vdash E_2 : \tau_3) \circ (x : \tau_1 \vdash E_1 : \tau_2) = (x : \tau_1 \vdash E_2[y \mapsto E_1] : \tau_3).$$

- The **monad** is given by $(T, \eta, (\cdot)^\dagger)$ where

$$\eta_\tau = (x : \tau \vdash [x]_T : T\tau)$$
$$(x : \tau_1 \vdash E : T\tau_2)^\dagger = (y : T\tau_1 \vdash \text{let } x \Leftarrow y \text{ in } E : T\tau_1).$$

- The **strength** of the monad is given by

$$\mathrm{st}_{\tau_1,\tau_2} = (x : \tau_1 \times T\tau_2 \vdash \text{let } z \Leftarrow \pi_2 x \text{ in } \langle \pi_1 x, z\rangle : T(\tau_1 \times \tau_2)).$$

Theorem 8. *$\mathcal{F}[\mathcal{S}]$ is indeed a CCC with a strong monad.* □

There is a natural interpretation ι of the monadic metalanguage in $\mathcal{F}[\mathcal{S}]$ with

$$\iota(\beta) = \beta \qquad \text{for } \beta \text{ in } \mathcal{S}_{type}$$
$$\iota(c) = (\;\vdash c : \tau) \qquad \text{for a constant } c \text{ of type } \tau \text{ in } \mathcal{S}_{const}.$$

extended to all types and objects as in Figure 3.3.

Theorem 9. *The interpretation ι of the monadic metalanguage in $\mathcal{F}[\mathcal{S}]$ is complete with respect to the equational theory described in Figure 3.2.*

Proof. As before, this holds by the definition of the quotienting. □

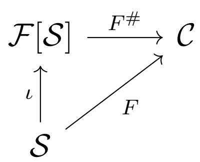
3.3.3 Free property

The free property of the interpretation of λ_{ml} is similar to that of the STLC.

Definition 21 (Strictly preserving a strong monad). *Give cartesian categories with strong monads* $(\mathcal{C}_1, T_1, \eta_1, (\cdot)^{\dagger_1}, \mathrm{st}_1)$ *and* $(\mathcal{C}_2, T_2, \eta_2, (\cdot)^{\dagger_2}, \mathrm{st}_2)$*, a functor* $F : \mathcal{C}_1 \to \mathcal{C}_2$ *is said to strictly preserve the strong monadic structure if for all objects* τ, τ_1, τ_2*,*

$$F(T_1(\tau)) = T_2(F(\tau))$$
$$F(\eta_{1,\tau}) = \eta_{2,F\tau}$$
$$F((f)^{\dagger_1}) = (F(f))^{\dagger_2}$$
$$F(\mathrm{st}_{1,\tau_1,\tau_2}) = \mathrm{st}_{2,F(\tau_1),F\tau_2}.$$
▲

Definition 22 (Free CCC with a strong monad over a signature). *Given a signature* $\mathcal{S} = (\mathcal{S}_{type}, \mathcal{S}_{const})$ *a CCC with a strong monad* $\mathcal{F}[\mathcal{S}]$ *is* free *over* $\mathcal{S}$ *iff there exists an interpretation* ι *of* $\mathcal{S}$ *in* $\mathcal{F}[\mathcal{S}]$ *such that for any CCC with a strong monad* $\mathcal{C}$*, and any interpretation* F *of* $\mathcal{S}$ *in* $\mathcal{C}$*, there is a unique strict CC-functor* F *strictly preserving the strong monadic structure, such that the following diagram commutes:*



i.e., for any $\beta \in \mathcal{S}_{type}$*,* $F^{\#}(\iota(\beta)) = F(\beta)$ *and any* $c \in \mathcal{S}_{const}$*,* $F^{\#}(\iota(c)) = F(c)$*.* ▲

Theorem 10. $\mathcal{F}[\mathcal{S}]$ *with* ι *is the free CCC with a strong monad over that signature.*

□

4

Computational lambda calculus and Freyd-categories

This chapter describes and proves the relationship between the semantics of computational lambda calculus and closed Freyd-categories. While it has been known that closed Freyd-categories provide sound and complete semantics of the computational lambda calculus, the detailed description of the interpretation and the syntactic closed Freyd-category, and the required proofs are original work. The key to the abbreviations used throughout this Chapter is available in Appendix A.

4.1 Computational lambda calculus

The computational lambda calculus is a simple programming language introduced by Moggi [18] as a generalization of the STLC that allows for modelling side-effecting computation. Its structure is similar to that of the STLC and λ_{ml}, but unlike the STLC, it is not necessarily pure, and unlike the λ_{ml}, side-effecting computations do not need to be treated differently in syntax from pure expressions.

Definition 23 (Signature for the computational lambda calculus)**.** *A* signature for the λ_{C} $\mathcal{S}$ *consists of:*

- *A set $\mathcal{S}_{type}$ of base types.*
- *A set $\mathcal{S}_{prim}$ describing the pure constants. $\mathcal{S}_{prim}$ has elements (c, τ) where c is the name of the constant and τ is a computational lambda calculus type.*
- *A set $\mathcal{S}_{efop}$ describing the effectful constants. $\mathcal{S}_{efop}$ has elements (c, τ) where c is the name of the constant and τ is a computational lambda calculus type.*

▲

The types of the computational lambda calculus are exactly the same as those of the STLC.

Definition 24 (Types of λ_{C})**.** *The types of the computational lambda calculus are given by the following grammar*

$$\tau ::= \beta \mid 1 \mid \tau_1 \times \tau_2 \mid \tau_1 \to \tau_2$$

where $\beta \in \mathcal{S}_{type}$ ranges over the given base types. ▲

We introduce a new concept: intuitively, *values* are terms that do not have side effects. Note that this definition differs from the definition of values often used for the untyped lambda calculus, where values are those terms that do not reduce. In this case, values also include complex values [12], terms that reduce, but do not have side effects, such as $\pi_1\langle (), x\rangle$.

Definition 25 (Terms of λ_{C})**.** *The terms of the computational lambda calculus are given by the following grammars of values and general computations*

$$V ::= x \mid c_{prim} \mid () \mid \pi_x V \mid \langle V_1, V_2 \rangle \mid let\ x \Leftarrow V_1\ in\ V_2 \mid \lambda x.M$$
$$M ::= c_{efop} \mid V \mid \pi_x M \mid \langle M_1, M_2 \rangle \mid let\ x \Leftarrow M_1\ in\ M_2 \mid M_1 M_2$$

where c_{prim} ranges over the given pure constant symbols, i.e., $(c_{prim}, \tau) \in \mathcal{S}_{prim}$ for some τ, and c_{efop} ranges over the given effectful constant symbols, i.e., $(c_{efop}, \tau) \in \mathcal{S}_{efop}$ for some τ. ▲

Note that an expression might not have side effects, and still might not be generated by the V grammar, for example, if it is of the form $(\lambda x.V_1)V_2$. This is not a problem as V is simply a helper construct and we will see later, that in that case, an equivalent expression (in the above case $V_2[x \mapsto V_1]$) might be generated by the value grammar.

Note also that in λ_{C}, every variable is pure, unlike in λ_{ml} where variables could have monadic types and hence correspond to potentially side-effecting computation.

The typing rules, described in Figure 4.1, are also similar to those of the STLC, with the exception of the (let) rule, and the rules for constants.

The equations of λ_{C} are described in Figure 4.2. These differ from the STLC ones in multiple points. In particular, in the η and β rules of products and functions, and in the (unit) rule, some of the terms are restricted to be values. The fnβ-rule is restricted to values to achieve a call-by-value semantics, i.e., a semantics where the values are computed before substituting them. Note that if a term does not have side effects (such as all the term of the STLC), while it makes a difference in the operational semantics whether we compute the term before or after substitution, it does not make a difference semantically. However, with side effects, it does, as it affects when and how many times the side effects are performed. Similarly, the

$$\frac{}{x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i}\ (\text{var})$$

$$\frac{(c, \tau) \in \mathcal{S}_{prim}}{\Gamma \vdash c : \tau}\ (\text{prim})$$

$$\frac{(c, \tau) \in \mathcal{S}_{efop}}{\Gamma \vdash c : \tau}\ (\text{efop})$$

$$\frac{}{\Gamma \vdash () : 1}\ (\text{unit})$$

$$\frac{\Gamma \vdash M : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i M : \tau_i}\ (\text{proj})$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \quad \Gamma \vdash M_2 : \tau_2}{\Gamma \vdash \langle M_1, M_2 \rangle : \tau_1 \times \tau_2}\ (\text{pair})$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash M_2 : \tau_2}{\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2 : \tau_2}\ (\text{let})$$

$$\frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x.M : \tau_1 \to \tau_2}\ (\text{abst})$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1 M_2 : \tau_2}\ (\text{app})$$

Figure 4.1: Typing rules of λ_{C}

prodβ specifies that we first compute the product, and only project out the correct position after it, and the letβ specifies that for a let-binding let $x \Leftarrow M_1$ in M_2 we want to compute M_1 before substituting it for x in M_2. We restrict certain terms in the η-rules to values to remain sound with respect to contextual equivalence.

The (letη), (letβ) and (assoc) rules are newly added as we have newly added the let construct. The (compproj), (comppair) and (compapp) rules are also new, intuitively, these describe how these constructs interact with side-effecting terms. Note that these equations also explicitly describe which order the terms are evaluated in. For example, the (comppair) rules specifies that $\langle M_1, M_2 \rangle$ should semantically agree with let $x \Leftarrow M_1$ in (let $y \Leftarrow M_2$ in $\langle x, y\rangle$), so these equations specify that the left position should be evaluated first in a product.

4.2 Freyd-categories

Freyd-categories were introduced as a model of languages with side effects [23]. Intuitively, premonoidal categories are better suited to modelling side effects than monoidal categories (such as cartesian categories), because they allow us to explicitly describe that side effects do not commute, as $(f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g)$ does not in general equal $(\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id})$. This is required as for example printing `hello` and then `world` should have different semantics than the other way around. Freyd-categories formalize the intuition that non-side-effecting expressions, i.e., values, do commute (so they can be modelled by a cartesian category $\mathbb{V}$), but computations, in general, satisfy a weaker structure (and form a premonoidal category $\mathbb{C}$ instead), and every value can be regarded as a general computation (so we have a structure-preserving functor $J : \mathbb{V} \to \mathbb{C}$).

Definition 26 (Binoidal category)**.** *A* binoidal category *is a category $\mathcal{C}$ together with:*

- *for any two of objects X, Y of $\mathcal{C}$, an object $X \otimes Y$ of $\mathcal{C}$*
- *for any object X, a functor $X \rtimes -$ such that for any object Y, $X \rtimes Y = X \otimes Y$*
- *for any object X, a functor $- \ltimes X$ such that for any object Y, $Y \ltimes X = Y \otimes X$.* ▲

For simplicity, for a morphism f we denote $X \rtimes f$ by $\mathrm{id} \otimes f$ when X is clear. Similarly, we also denote $f \ltimes X$ by $f \otimes \mathrm{id}$.

$$\frac{\Gamma \vdash V : \tau_1 \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \text{let } x \Leftarrow V \text{ in } M \equiv M[x \mapsto V] : \tau_2} \quad (\text{let}\beta)$$

$$\frac{\Gamma \vdash V_1 : \tau_1 \quad \Gamma \vdash V_2 : \tau_2 \quad i \in \{1, 2\}}{\Gamma \vdash \pi_i \langle V_1, V_2 \rangle \equiv V_i : \tau_i} \quad (\text{prod}\beta)$$

$$\frac{\Gamma \vdash V : \tau_2 \quad \Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash (\lambda x.M)V \equiv M[x \mapsto V] : \tau_2} \quad (\text{fn}\beta)$$

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash \text{let } x \Leftarrow M \text{ in } x \equiv M : \tau} \quad (\text{let}\eta)$$

$$\frac{\Gamma \vdash V : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 V, \pi_2 V \rangle \equiv V : \tau_1 \times \tau_2} \quad (\text{prod}\eta)$$

$$\frac{\Gamma \vdash V : \tau_1 \to \tau_2 \quad x \text{ is not free in } V}{\Gamma \vdash \lambda x.Vx \equiv V : \tau_1 \to \tau_2} \quad (\text{fn}\eta)$$

$$\frac{\Gamma \vdash E_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash E_2 : \tau_2 \quad \Gamma, y : \tau_2 \vdash E_3 : \tau_3}{\begin{array}{l}\Gamma \vdash \text{let } y \Leftarrow (\text{let } x \Leftarrow E_1 \text{ in } E_2) \text{ in } E_3 \equiv \\ \qquad \text{let } x \Leftarrow E_1 \text{ in } (\text{let } y \Leftarrow E_2 \text{ in } E_3) : \tau_3\end{array}} \quad (\text{assoc})$$

$$\frac{\Gamma \vdash V : 1}{\Gamma \vdash () \equiv V : 1} \quad (\text{unit})$$

$$\frac{\Gamma \vdash M : \tau_1 \times \tau_2 \quad i \in \{1, 2\}}{\Gamma \vdash \pi_i M \equiv \text{let } x \Leftarrow M \text{ in } \pi_i x : \tau_i} \quad (\text{compproj})$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \quad \Gamma \vdash M_2 : \tau_2}{\begin{array}{l}\Gamma \vdash \langle M_1, M_2 \rangle \equiv \\ \qquad \text{let } x \Leftarrow M_1 \text{ in } (\text{let } y \Leftarrow M_2 \text{ in } \langle x, y \rangle) : \tau_1 \times \tau_2\end{array}} \quad (\text{comppair})$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1 M_2 \equiv \text{let } x \Leftarrow M_1 \text{ in } (\text{let } y \Leftarrow M_2 \text{ in } xy) : \tau_2} \quad (\text{compapp})$$

Together with the equivalence relation rules (reflexivity, symmetry, transitivity), and congruence rules for each constructor.

Figure 4.2: Equations of λ_{C}

Definition 27 (Central morphism)**.** *In a binoidal category $\mathcal{C}$, $f : X_1 \to Y_1$ is a* central morphism *iff for any morphism $g : X_2 \to Y_2$ the following diagrams commute:*

$$\begin{CD} X_1 \otimes X_2 @>{X_1 \rtimes g}>> X_1 \otimes Y_2 \\ @V{f \ltimes X_2}VV @VV{f \ltimes Y_2}V \\ Y_1 \otimes X_2 @>>{Y_1 \rtimes g}> Y_1 \otimes Y_2 \end{CD}$$

and

$$\begin{CD} X_2 \otimes X_1 @>{g \ltimes X_1}>> Y_2 \otimes X_1 \\ @V{X_2 \rtimes f}VV @VV{Y_2 \rtimes f}V \\ X_2 \otimes Y_1 @>>{g \ltimes Y_1}> Y_2 \otimes Y_1. \end{CD}$$

▲

Definition 28 (Premonoidal category)**.** *A* premonoidal category *is a category $\mathcal{C}$ with:*

- *an object I*
- *a natural transformation a with components*

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

where all components are isomorphisms and central

- *natural transformations λ and ρ with components*

$$\lambda_X : X \otimes I \to X$$

$$\rho_X : I \otimes X \to X$$

where all components are isomorphisms and central, such that the pentagon *law and triangle law holds, i.e., the following two diagrams commute:*

$$
\begin{array}{ccc}
(X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\
& {\scriptstyle \lambda_X \ltimes Y} \searrow \quad \swarrow {\scriptstyle X \rtimes \rho_Y} & \\
& X \otimes Y &
\end{array}
$$

and

$$
\begin{array}{ccc}
((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a_{W \otimes X,Y,Z}} & (W \otimes X) \otimes (Y \otimes Z) \\
{\scriptstyle a_{W,X,Y} \ltimes Z} \downarrow & & \\
(W \otimes (X \otimes Y)) \otimes Z & & \downarrow {\scriptstyle a_{W,X,Y \otimes Z}} \\
{\scriptstyle a_{W,X \otimes Y,Z}} \downarrow & & \\
W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow[W \rtimes a_{X,Y,Z}]{} & W \otimes (X \otimes (Y \otimes Z))
\end{array}
$$

▲

Example 2. *Every monoidal category is also a premonoidal category.* ▲

Definition 29 (Symmetry of a premonoidal category)**.** *A* symmetry *of a premonoidal category $\mathcal{C}$ is a central natural isomorphism with components*

$$s_{X,Y} : X \otimes Y \to Y \otimes X$$

such that for any X, Y, $s_{Y,X} \circ s_{X,Y} = \mathrm{id}_{X \otimes Y}$ and the following diagram commutes:

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{s_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\
{\scriptstyle s_{X,Y} \otimes \mathrm{id}} \downarrow & & & & \downarrow {\scriptstyle a_{Y,Z,X}} \\
(Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[\mathrm{id} \otimes s_{X,Z}]{} & Y \otimes (Z \otimes X)
\end{array}
$$

▲

Definition 30 (Freyd category)**.** *A* Freyd category *[20] is $\mathbb{V} \xrightarrow{J} \mathbb{C}$ where:*

- $\mathbb{V}$ *is a category with finite products*

- *$\mathbb{C}$ is a symmetric premonoidal category*
- *$\mathbb{V}$ and $\mathbb{C}$ have the same objects*
- *$J : \mathbb{V} \to \mathbb{C}$ is an identity-on-object functor that strictly preserves symmetric premonoidal structure and maps every morphism of $\mathbb{V}$ to a central morphism in $\mathbb{C}$.* ▲

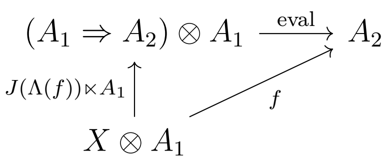
Definition 31 (Closed Freyd category)**.** *A Freyd category $\mathbb{V} \xrightarrow{J} \mathbb{C}$ is* closed *if for every object X, the functor $J(- \times X) : \mathbb{V} \to \mathbb{C}$ has a right adjoint.*

Explicitly, if we denote the right adjoint by $(X \Rightarrow -)$, we get

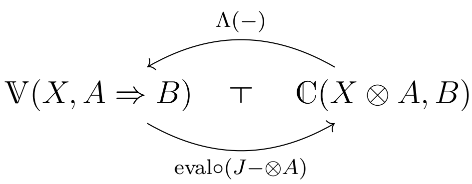
$$\mathbb{C}\Big(J(A_1 \times X), A_2\Big) \cong \mathbb{V}(A_2, X \Rightarrow A_1),$$

natural in A_1 and A_2. ▲

So if we denote the counit of this adjunction by eval, one has that for any $f : X \times A_1 \to A_2$ in $\mathbb{C}$, there is a unique $\Lambda(f) : X \to (A_1 \Rightarrow A_2)$ in $\mathbb{V}$ such that the following diagram commutes:



With this notation, we might also represent the above adjunction as:



This adjunction gives the following η and β rules of exponentials in closed Freyd-categories:

- For any $f : X \to (A \Rightarrow B)$ in $\mathbb{V}$

$$\Lambda(\text{eval} \circ (Jf \otimes A)) \stackrel{\eta}{=} f.$$

- For any $f : X \otimes A \to B$ in $\mathbb{C}$

$$\text{eval} \circ (J\Lambda(f) \otimes A) \stackrel{\beta}{=} f.$$

These have a similar form to the (fnη) $\lambda x.Vx \equiv V$ and (fnβ) $\lambda x.MV \equiv M[x \mapsto V]$ rules for the computational lambda calculus, and as we will see later, are indeed closely related.

Example 3. *For a cartesian category with a strong monad $(\mathcal{C}, T, \eta, (\cdot)^\dagger)$, and $\mathcal{C}_T$ the Kleisli-category of $\mathcal{C}$ with the monad T, $\mathcal{C} \xrightarrow{\eta \circ -} \mathcal{C}_T$ is a Freyd-category, and it is closed iff $\mathcal{C}$ has Kleisli-exponentials [13].* ▲

4.3 Interpretation of λ_{C} in a closed Freyd-category

Definition 32 (Interpretation of a signature in a Freyd-category)**.** *Given a signature $\mathcal{S} = (\mathcal{S}_{type}, \mathcal{S}_{prim}, \mathcal{S}_{efop})$, and a Freyd-category $\mathbb{V} \xrightarrow{J} \mathbb{C}$, an* interpretation *of $\mathcal{S}$ in $\mathcal{C}$ is a map $i_{type} : \mathcal{S}_{type} \to ob(\mathbb{V})$ extended to a mapping of all types to objects as in Figure 4.3, and maps i_{prim}, i_{efop} that map $(c, \tau) \in \mathcal{S}_{prim}$ to a morphism $1 \to [\![\tau]\!]$ in $\mathbb{V}$ and $(c, \tau) \in \mathcal{S}_{efop}$ to a morphism $1 \to [\![\tau]\!]$ in $\mathbb{C}$ respectively, that is extended to a mapping from all values and terms respectively, as in Figure 4.3.* ▲

Note that as before, types are interpreted as an object of $\mathbb{V}$ and $\mathbb{C}$, denoted by $[\![\tau]\!] = [\![\tau]\!]_{\mathbb{V}} = [\![\tau]\!]_{\mathbb{C}}$. Furthermore, every well-typed term $\Gamma \vdash M : \tau$ has an interpretation in $\mathbb{C}$ given by $[\![\Gamma \vdash M : \tau]\!]_{\mathbb{C}}$. Values $\Gamma \vdash V : \tau$ also have an interpretation in $\mathbb{V}$ given by $[\![\Gamma \vdash V : \tau]\!]_{\mathbb{V}}$.

Notes on notation. Where the type of a variable in a context or the type of a term is deducible from context, it is omitted for brevity. E.g., we might use

$$[\![\Gamma, x \vdash M]\!]_{\mathrm{C}}$$

as a shorthand for $[\![\Gamma, x : \tau_1 \vdash M : \tau_2]\!]_{\mathrm{C}}$.

4.4 Soundness

In this section, we prove that the interpretation of λ_{C} is sound with respect to the equations of λ_{C}. The two key lemmas required to prove this are the Weakening lemma (Lemma 3) and the Substitution lemma (Lemma 4) from below.

Lemma 1. *In a closed Freyd-category $\mathbb{V} \xrightarrow{J} \mathbb{C}$, for objects X, A, B, and morphisms $f, g : X \to A \Rightarrow B$ in $\mathbb{V}$,*

$$f = g \iff (\mathrm{eval} \circ (Jf \otimes A)) = (\mathrm{eval} \circ (Jg \otimes A)).$$

Proof. The $\Rightarrow$ direction holds trivially.

To see the $\Leftarrow$ direction, note that $f \stackrel{\eta}{=} \Lambda(\mathrm{eval} \circ (Jf \otimes A))$, and similarly $\Lambda(\mathrm{eval} \circ (Jg \otimes A)) \stackrel{\eta}{=} g$. □

Lemma 2. *In a closed Freyd-category $\mathbb{V} \xrightarrow{J} \mathbb{C}$, for objects X, X', A, B, and morphisms $f : X' \otimes A \to B$ in $\mathbb{C}$ and $g : X \to X'$ in $\mathbb{V}$,*

$$\Lambda(f \circ (Jg \otimes A)) = \Lambda(f) \circ g.$$

$$
\begin{aligned}
[\![\beta]\!] &= i_{type}(\beta) \\
[\![1]\!] &= 1 \\
[\![\sigma_1 \times \sigma_2]\!] &= [\![\sigma_1]\!] \times [\![\sigma_2]\!] \\
[\![\sigma_1 \to \sigma_2]\!] &= [\![\sigma_1]\!] \Rightarrow [\![\sigma_2]\!]
\end{aligned}
$$

$$
\begin{aligned}
[\![\Diamond]\!] &= 1 \\
[\![x_1 : \tau_1, \dots x_n : \tau_n]\!] &= (([\![\tau_1]\!] \times [\![\tau_2]\!]) \times \dots) \times [\![\tau_n]\!]
\end{aligned}
$$

$$
\begin{aligned}
[\![\Gamma \vdash c_{prim} : \tau]\!]_{\mathbb{V}} &= i_{prim}(c_{prim}, \tau) \circ ! \\
[\![\Gamma \vdash x_i : \sigma_i]\!]_{\mathbb{V}} &= \left([\![\Gamma]\!] \xrightarrow{\pi_i} [\![\sigma_i]\!]\right) \\
[\![\Gamma \vdash () : 1]\!]_{\mathbb{V}} &= \left([\![\Gamma]\!] \xrightarrow{!} [\![\sigma_i]\!]\right) \\
[\![\Gamma \vdash \lambda x.M : \sigma_1 \to \sigma_2]\!]_{\mathbb{V}} &= \Lambda\left([\![\Gamma]\!] \otimes [\![\sigma_1]\!] \xrightarrow{[\![\Gamma, x:\sigma_1 \vdash M]\!]_{\mathbb{C}}} [\![\sigma_2]\!]\right) \\
[\![\Gamma \vdash \langle V_1, V_2\rangle : \sigma_1 \times \sigma_2]\!]_{\mathbb{V}} &= \left([\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash V_1]\!]_{\mathbb{V}}, [\![\Gamma \vdash V_2]\!]_{\mathbb{V}}\rangle} [\![\Gamma]\!]\right) \\
[\![\Gamma \vdash \pi_i V : \sigma_i]\!]_{\mathbb{V}} &= \left([\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash V]\!]_{\mathbb{V}}} [\![\sigma_1]\!] \times [\![\sigma_2]\!] \xrightarrow{\pi_i} [\![\sigma_i]\!]\right) \\
[\![\Gamma \vdash \text{let } x \Leftarrow V_1 \text{ in } V_2 : \sigma_2]\!]_{\mathbb{V}} &= \left([\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}, [\![\Gamma \vdash V_1]\!]_{\mathbb{V}}\rangle} [\![\Gamma]\!] \times [\![\sigma_1]\!] \xrightarrow{[\![\Gamma, x:\sigma_1 \vdash V_2]\!]_{\mathbb{V}}} [\![\sigma_2]\!]\right)
\end{aligned}
$$

$$
\begin{aligned}
[\![\Gamma \vdash c_{efop} : \tau]\!]_{\mathbb{C}} &= i_{efop}(c_{efop}, \tau) \circ J! \\
[\![\Gamma \vdash \langle M_1, M_2\rangle : \sigma_1 \times \sigma_2]\!]_{\mathbb{C}} &= ([\![\Gamma]\!] \rtimes [\![\Gamma \vdash M_2 : \sigma_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1 : \sigma_1]\!]_{\mathbb{C}} \ltimes [\![\Gamma]\!]) \\
&\quad \circ J\Delta \\
[\![\Gamma \vdash \pi_i M : \sigma_i]\!]_{\mathbb{C}} &= J\pi_i \circ [\![\Gamma \vdash M : \sigma_1 \times \sigma_2]\!]_{\mathbb{C}} \\
[\![\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2 : \sigma_2]\!]_{\mathbb{C}} &= [\![\Gamma, x : \sigma_1 \vdash M_2 : \sigma_2]\!]_{\mathbb{C}} \circ ([\![\Gamma]\!] \rtimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
[\![\Gamma \vdash M_1 M_2 : \sigma_2]\!]_{\mathbb{C}} &= \mathrm{eval} \circ ([\![\sigma_1 \to \sigma_2]\!] \rtimes [\![\Gamma \vdash M_2 : \sigma_2]\!]_{\mathbb{C}}) \\
&\quad \circ ([\![\Gamma \vdash M_1 : \sigma_1 \to \sigma_2]\!]_{\mathbb{C}} \ltimes [\![\Gamma]\!]) \circ J\Delta
\end{aligned}
$$

Figure 4.3: Interpretation λ_C in a closed Freyd-category

Proof. Using Lemma 1, it is sufficient to show

$$\mathrm{eval} \circ (J(\Lambda(f \circ (Jg \otimes A))) \otimes A) = \mathrm{eval} \circ (J(\Lambda(f) \circ g) \otimes A).$$

But indeed, $\mathrm{eval} \circ (J(\Lambda(f \circ (Jg \otimes A))) \otimes A) \stackrel{\beta}{=} f \circ (Jg \otimes A)$ and using that J and $- \otimes A$ are functors:

$$\begin{aligned} &\mathrm{eval} \circ (J(\Lambda(f) \circ g) \otimes A) \\ &\quad = \mathrm{eval} \circ ((J(\Lambda(f)) \circ Jg) \otimes A) \\ &\quad = \mathrm{eval} \circ (J(\Lambda(f)) \otimes A) \circ (Jg \otimes A) \\ &\quad \stackrel{\beta}{=} f \circ (Jg \otimes A). \end{aligned}$$

□

Lemma 3 (Weakening). *For contexts $\Gamma_1 = x_1 : \tau_1, \ldots x_n : \tau_n$, $\Gamma_2 = y_1 : \sigma_1, \ldots, y_m : \sigma_m$, define $\rho : \Gamma_1 \to \Gamma_2$ to be a* context renaming *if, for each x_i in Γ_1, $\rho(x_i) = y_j$ for y_j is in Γ_2 and $\tau_i = \sigma_j$. Furthermore, define ρ_i to be the (unique) index j such that $\rho(x_i) = y_j$.*

Now for $[\![-]\!]_{\mathbb{V}}$, $[\![-]\!]_{\mathbb{C}}$ the interpretation of λ_{C} in the closed Freyd-category $\mathbb{V} \xrightarrow{J} \mathbb{C}$, for any value $\Gamma_1 \vdash V : \tau$,

$$[\![\Gamma_1 \vdash V : \tau]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle = [\![\Gamma_2 \vdash V[x_i \mapsto \rho(x_i)] : \tau]\!]_{\mathbb{V}}$$

in $\mathbb{V}$, and for any term $\Gamma_1 \vdash M : \tau$,

$$[\![\Gamma_1 \vdash M : \tau]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle = [\![\Gamma_2 \vdash M[x_i \mapsto \rho(x_i)] : \tau]\!]_{\mathbb{C}}$$

in $\mathbb{C}$.

Proof. We are going to prove this statement by structural induction on the grammar of values and computations. For each term, we have to show that the denotation of the right-hand side agrees with the denotation of the left-hand side.

- **Case: var**

$$\begin{aligned}
&[\![\Gamma_2 \vdash x_j[x_i \mapsto \rho_i(x_i)]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma_2 \vdash \rho(x_j)]\!]_{\mathbb{V}} \\
&\quad = \pi_{\rho_j} \\
&\quad = \pi_j \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = [\![\Gamma_1 \vdash x_j]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}$$

- **Case: prim**

$$\begin{aligned}
&[\![\Gamma \vdash c[x_i \mapsto \rho_i(x_i)] : \tau]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma \vdash c : \tau]\!]_{\mathbb{V}} \\
&\quad = [\![\ \vdash c : \tau]\!]_{\mathbb{V}} \circ ! \\
&\quad = [\![\ \vdash c : \tau]\!]_{\mathbb{V}} \circ ! \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash c : \tau]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}$$

- **Case: unit**

$$[\![\Gamma_1 \vdash () : 1]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle = [\![\Gamma_2 \vdash ()[x_i \mapsto \rho(x_i)] : 1]\!]_{\mathbb{V}}$$

because both are morphisms from the $[\![\Gamma_2]\!]$ object to the terminal object.

- **Case: val-proj**

$$[\![\Gamma_2 \vdash (\pi_j V)[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}}$$

$$
\begin{aligned}
&= [\![\Gamma_2 \vdash \pi_j(V[x_i \mapsto \rho(x_i)])]\!]_{\mathbb{V}} \\
&= \pi_j \circ [\![\Gamma_2 \vdash V[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}} \\
&\overset{\text{IH}}{=} \pi_j \circ [\![\Gamma_1 \vdash V]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&= [\![\Gamma_1 \vdash (\pi_j V)]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

- **Case: val-pair**

$$
\begin{aligned}
&[\![\Gamma_2 \vdash \langle V_1, V_2 \rangle [x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma_2 \vdash \langle V_1[x_i \mapsto \rho(x_i)], V_2[x_i \mapsto \rho(x_i)] \rangle]\!]_{\mathbb{V}} \\
&\quad = \langle [\![\Gamma_2 \vdash V_1[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}}, [\![\Gamma_2 \vdash V_2[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}} \rangle \\
&\quad \overset{\text{IH}}{=} \langle [\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle, [\![\Gamma_1 \vdash V_2]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \rangle \\
&\quad = \langle [\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}}, [\![\Gamma_1 \vdash V_2]\!]_{\mathbb{V}} \rangle \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = [\![\Gamma_1 \vdash \langle V_1, V_2 \rangle]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

- **Case: val-let**

$$
\begin{aligned}
&[\![\Gamma_2 \vdash (\text{let } x \Leftarrow V_1 \text{ in } V_2)[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma_2 \vdash \text{let } x \Leftarrow V_1[x_i \mapsto \rho(x_i)] \text{ in } V_2[x_i \mapsto \rho(x_i), x \mapsto x]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma_2, x \vdash V_2[x_i \mapsto \rho(x_i), x \mapsto x]]\!]_{\mathbb{V}} \circ (\text{id} \times [\![\Gamma_2 \vdash V_1[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}}) \circ \Delta \\
&\quad \overset{\text{IH}}{=} [\![\Gamma_1, x \vdash V_2]\!]_{\mathbb{V}} \circ \langle \pi_{\rho'_1}, \ldots, \pi_{\rho'_n}, \pi_{\rho'_{n+1}} \rangle \\
&\qquad\quad \circ (\text{id} \times ([\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \circ \Delta \\
&\quad = [\![\Gamma_1, x \vdash V_2]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1} \circ \pi_1, \ldots, \pi_{\rho_n} \circ \pi_1, \pi_2 \rangle \\
&\qquad\quad \circ (\text{id} \times ([\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \circ \Delta \\
&\quad = [\![\Gamma_1, x \vdash V_2]\!]_{\mathbb{V}} \circ (\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \times \text{id}) \circ (\text{id} \times [\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}}) \\
&\qquad\quad \circ (\text{id} \times \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \circ \Delta
\end{aligned}
$$

$$= [\![\Gamma_1, x \vdash V_2]\!]_{\mathbb{V}} \circ (\mathrm{id} \times [\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}}) \circ (\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \times \mathrm{id})$$

$$\circ\, (\mathrm{id} \times \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \circ \Delta$$

$$= [\![\Gamma_1, x \vdash V_2]\!]_{\mathbb{V}} \circ (\mathrm{id} \times [\![\Gamma_1 \vdash V_1]\!]_{\mathbb{V}}) \circ \Delta \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle$$

$$= [\![\Gamma_1 \vdash \text{let } x \Leftarrow V_1 \text{ in } V_2]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle$$

- **Case: abst**

$$[\![\Gamma_2 \vdash (\lambda x.M)[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}}$$

$$= [\![\Gamma_2 \vdash \lambda x.(M[x_i \mapsto \rho(x_i), x \mapsto x])]\!]_{\mathbb{V}}$$

$$= \Lambda\Big([\![\Gamma_2, x \vdash M[x_i \mapsto \rho(x_i), x \mapsto x]]\!]_{\mathbb{C}}\Big)$$

$$\overset{\text{IH}}{=} \Lambda\Big([\![\Gamma_1, x \vdash M]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho'_1}, \ldots, \pi_{\rho'_n}, \pi_{\rho'_{n+1}} \rangle\Big)$$

$$= \Lambda\Big([\![\Gamma_1, x \vdash M]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1} \circ \pi_1, \ldots, \pi_{\rho_n} \circ \pi_1, \pi_2 \rangle\Big)$$

$$= \Lambda\Big([\![\Gamma_1, x \vdash M]\!]_{\mathbb{C}} \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id})\Big)$$

$$\overset{*}{=} \Lambda\Big([\![\Gamma_1, x \vdash M]\!]_{\mathbb{C}}\Big) \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle$$

$$= [\![\Gamma_1 \vdash \lambda x.M]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle$$

where $*$ hold by Lemma 2.

- **Case: val-to-comp**

$$[\![\Gamma_2 \vdash V[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}}$$

$$= J[\![\Gamma_2 \vdash V[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{V}}$$

$$\overset{\text{IH}}{=} J([\![\Gamma_1 \vdash V]\!]_{\mathbb{V}} \circ \langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)$$

$$= (J[\![\Gamma_1 \vdash V]\!]_{\mathbb{V}}) \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle$$

$$= [\![\Gamma_1 \vdash V]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle$$

- **Case: efop**

$$
\begin{aligned}
&[\![\Gamma \vdash c[x_i \mapsto \rho(x_i)] : \tau]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma \vdash c : \tau]\!]_{\mathbb{C}} \\
&\quad = [\![\ \vdash c : \tau]\!]_{\mathbb{C}} \circ J! \\
&\quad = [\![\ \vdash c : \tau]\!]_{\mathbb{C}} \circ J! \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash c : \tau]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

- **Case: comp-proj**

$$
\begin{aligned}
&[\![\Gamma_2 \vdash (\pi_j M)[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma_2 \vdash \pi_j(M[x_i \mapsto \rho(x_i)])]\!]_{\mathbb{C}} \\
&\quad = J\pi_j \circ [\![\Gamma_2 \vdash M[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \\
&\quad = J\pi_j \circ [\![\Gamma_2 \vdash M[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \\
&\quad \overset{\text{IH}}{=} J\pi_j \circ ([\![\Gamma_1 \vdash M]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \\
&\quad = J\pi_j \circ [\![\Gamma_1 \vdash M]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = [\![\Gamma_1 \vdash \pi_j M]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

- **Case: comp-pair**

$$
\begin{aligned}
&[\![\Gamma_2 \vdash \langle M_1, M_2 \rangle[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma_2 \vdash \langle M_1[x_i \mapsto \rho(x_i)], M_2[x_i \mapsto \rho(x_i)] \rangle]\!]_{\mathbb{C}} \\
&\quad = (\mathrm{id} \otimes [\![\Gamma_2 \vdash M_2[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}}) \circ ([\![\Gamma_2 \vdash M_1[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \\
&\quad \overset{\text{IH}}{=} (\mathrm{id} \otimes ([\![\Gamma_1 \vdash M_2]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \\
&\qquad \circ (([\![\Gamma_1 \vdash M_1]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \otimes \mathrm{id}) \circ J\Delta
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathrm{id} \otimes (\llbracket \Gamma_1 \vdash M_2 \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \\
&\qquad \circ ((\llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \otimes \mathrm{id}) \circ J\Delta \\
&= (\mathrm{id} \otimes \llbracket \Gamma_1 \vdash M_2 \rrbracket_{\mathbb{C}}) \circ (\mathrm{id} \otimes J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \\
&\qquad \circ (\llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}} \otimes \mathrm{id}) \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id}) \circ J\Delta \\
&= (\mathrm{id} \otimes \llbracket \Gamma_1 \vdash M_2 \rrbracket_{\mathbb{C}}) \circ (\llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}} \otimes \mathrm{id}) \\
&\qquad \circ (\mathrm{id} \otimes J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id}) \circ J\Delta \\
&= (\mathrm{id} \otimes \llbracket \Gamma_1 \vdash M_2 \rrbracket_{\mathbb{C}}) \circ (\llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&= \llbracket \Gamma_1 \vdash \langle M_1, M_2 \rangle \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

- **Case: comp-let**

$$
\begin{aligned}
&\llbracket \Gamma_2 \vdash (\mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ M_2)[x_i \mapsto \rho(x_i)] \rrbracket_{\mathbb{C}} \\
&\quad = \llbracket \Gamma_2 \vdash \mathrm{let}\ x \Leftarrow (M_1[x_i \mapsto \rho(x_i)])\ \mathrm{in}\ (M_2[x_i \mapsto \rho(x_i), x \mapsto x]) \rrbracket_{\mathbb{C}} \\
&\quad = \llbracket \Gamma_2, x \vdash (M_2[x_i \mapsto \rho(x_i), x \mapsto x]) \rrbracket_{\mathbb{C}} \\
&\qquad\quad \circ (\mathrm{id} \otimes \llbracket \Gamma_2 \vdash M_1[x_i \mapsto \rho(x_i)] \rrbracket_{\mathbb{C}}) \circ J\Delta \\
&\quad \overset{\mathrm{IH}}{=} \llbracket \Gamma_1, x \vdash M_2 \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1} \circ \pi_1, \ldots, \pi_{\rho_n} \circ \pi_1, \pi_2 \rangle \\
&\qquad\quad \circ (\mathrm{id} \otimes (\llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \circ J\Delta \\
&\quad = \llbracket \Gamma_1, x \vdash M_2 \rrbracket_{\mathbb{C}} \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id}) \\
&\qquad\quad \circ (\mathrm{id} \otimes (\llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \circ J\Delta \\
&\quad = \llbracket \Gamma_1, x \vdash M_2 \rrbracket_{\mathbb{C}} \circ (\mathrm{id} \otimes \llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}}) \\
&\qquad\quad \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id}) \circ (\mathrm{id} \otimes J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \circ J\Delta \\
&\quad = \llbracket \Gamma_1, x \vdash M_2 \rrbracket_{\mathbb{C}} \circ (\mathrm{id} \otimes \llbracket \Gamma_1 \vdash M_1 \rrbracket_{\mathbb{C}}) \circ J\Delta \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = \llbracket \Gamma_1 \vdash \mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ M_2 \rrbracket_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

- **Case: app**

$$
\begin{aligned}
&[\![\Gamma_2 \vdash (M_1 M_2)[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma_2 \vdash (M_1[x_i \mapsto \rho(x_i)])(M_2[x_i \mapsto \rho(x_i)])]\!]_{\mathbb{C}} \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma_2 \vdash M_2[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}}) \\
&\qquad\quad \circ ([\![\Gamma_2 \vdash M_1[x_i \mapsto \rho(x_i)]]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \\
&\quad \overset{\mathrm{IH}}{=} \mathrm{eval} \circ (\mathrm{id} \otimes ([\![\Gamma_1 \vdash M_2]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \\
&\qquad\quad \circ (([\![\Gamma_1 \vdash M_1]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle) \otimes \mathrm{id}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma_1 \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \circ ([\![\Gamma_1 \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \\
&\qquad\quad \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma_1 \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma_1 \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle)) \\
&\qquad\quad \circ (J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \otimes \mathrm{id}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma_1 \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma_1 \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle \\
&\quad = [\![\Gamma_1 \vdash M_1 M_2]\!]_{\mathbb{C}} \circ J\langle \pi_{\rho_1}, \ldots, \pi_{\rho_n} \rangle
\end{aligned}
$$

□

Lemma 4 (Substitution lemma)**.** *For $[\![-]\!]_{\mathbb{V}}$, $[\![-]\!]_{\mathbb{C}}$ the interpretation of λ_{C} in the closed Freyd-category $\mathbb{V} \xrightarrow{J} \mathbb{C}$, for any values $\Gamma \vdash U_i : \tau_i$ for $i = 1, \ldots, n$, and any value $x_1 : \tau_1, \ldots, x_n : \tau_n \vdash V : \tau$*

$$[\![\Gamma \vdash V[x_i \mapsto U_i] : \tau]\!]_{\mathbb{V}} = [\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash V : \tau]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle$$

in $\mathbb{V}$, and for any term $x_1 : \tau_1, \ldots, x_n : \tau_n \vdash M : \tau$,

$$[\![\Gamma \vdash M[x_i \mapsto U_i] : \tau]\!]_{\mathbb{C}} = [\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash M : \tau]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle$$

in $\mathbb{C}$.

Proof. As above, we are going to prove this statement by structural induction on the grammar of values and computations.

- **Case: var**

$$
\begin{aligned}
&[\![\Gamma \vdash x_j[x_i \mapsto U_i]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma \vdash U_j]\!]_{\mathbb{V}} \\
&\quad = \pi_j \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash x_j]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}
$$

- **Case: prim**

$$
\begin{aligned}
&[\![\Gamma \vdash c[x_i \mapsto U_i] : \tau]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma \vdash c : \tau]\!]_{\mathbb{V}} \\
&\quad = [\![\vdash c : \tau]\!]_{\mathbb{V}} \circ ! \\
&\quad = [\![\vdash c : \tau]\!]_{\mathbb{V}} \circ ! \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash c : \tau]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}
$$

- **Case: unit**

$$
\begin{aligned}
&[\![\Gamma \vdash ()[x_i \mapsto U_i]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma \vdash ()]\!]_{\mathbb{V}} \\
&\quad =! \\
&\quad =! \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash ()]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}
$$

- **Case: val-proj**

$$\begin{aligned}
&[\![\Gamma \vdash (\pi_j V)[x_i \mapsto U_i]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma \vdash \pi_j(V[x_i \mapsto U_i]]\!]_{\mathbb{V}}) \\
&\quad = \pi_j \circ [\![\Gamma \vdash (V[x_i \mapsto U_i])]\!]_{\mathbb{V}} \\
&\quad \overset{\text{IH}}{=} \pi_j \circ [\![x_1, \ldots, x_n \vdash V]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash \pi_j V]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}$$

- **Case: val-pair**

$$\begin{aligned}
&[\![\Gamma \vdash \langle V_1, V_2 \rangle [x_i \mapsto U_i]]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma \vdash \langle V_1[x_i \mapsto U_i], V_2[x_i \mapsto U_i] \rangle]\!]_{\mathbb{V}} \\
&\quad = \langle [\![\Gamma \vdash V_1[x_i \mapsto U_i]]\!]_{\mathbb{V}}, [\![\Gamma \vdash V_2[x_i \mapsto U_i]]\!]_{\mathbb{V}} \rangle \\
&\quad \overset{\text{IH}}{=} \langle [\![x_1, \ldots, x_n \vdash \pi_j V_1]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle, \\
&\qquad\quad [\![x_1, \ldots, x_n \vdash \pi_j V_2]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \rangle \\
&\quad = \langle [\![x_1, \ldots, x_n \vdash \pi_j V_1]\!]_{\mathbb{V}}, [\![x_1, \ldots, x_n \vdash \pi_j V_2]\!]_{\mathbb{V}} \rangle \\
&\qquad\quad \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash \langle V_1, V_2 \rangle]\!]_{\mathbb{V}} \circ \langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}$$

- **Case: val-let**

$$\begin{aligned}
&[\![\Gamma \vdash (\text{let } x \Leftarrow V_1 \text{ in } V_2)[x_i \mapsto U_i]]\!]_{\mathbb{V}} = \\
&\quad = [\![\Gamma \vdash (\text{let } x \Leftarrow (V_1[x_i \mapsto U_i]) \text{ in } (V_2[x_i \mapsto U_i, x \mapsto x]))]\!]_{\mathbb{V}} \\
&\quad = [\![\Gamma, x \vdash V_2[x_i \mapsto U_i, x \mapsto x]]\!]_{\mathbb{V}} \circ (\text{id} \times [\![\Gamma \vdash V_1[x_i \mapsto U_i]]\!]_{\mathbb{V}}) \circ \Delta \\
&\quad \overset{\text{IH}}{=} [\![x_1, \ldots x_n, x \vdash V_2]\!]_{\mathbb{V}} \circ \langle [\![\Gamma, x \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma, x \vdash U_n]\!]_{\mathbb{V}}, [\![\Gamma, x \vdash x]\!]_{\mathbb{V}} \rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad \circ (\mathrm{id} \times (\llbracket x_1, \ldots, x_n \vdash V_1 \rrbracket_{\mathbb{V}} \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle)) \circ \Delta \\
&\stackrel{\mathrm{w}}{=} \llbracket x_1, \ldots x_n, x \vdash V_2 \rrbracket_{\mathbb{V}} \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}} \circ \pi_1, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \circ \pi_1, \pi_2 \rangle \\
&\quad \circ (\mathrm{id} \times (\llbracket x_1, \ldots, x_n \vdash V_1 \rrbracket_{\mathbb{V}} \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle)) \circ \Delta \\
&= \llbracket x_1, \ldots x_n, x \vdash V_2 \rrbracket_{\mathbb{V}} \circ (\langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle \times \mathrm{id}) \\
&\quad \circ (\mathrm{id} \times \llbracket x_1, \ldots, x_n \vdash V_1 \rrbracket_{\mathbb{V}}) \circ (\mathrm{id} \times \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle) \circ \Delta \\
&= \llbracket x_1, \ldots x_n, x \vdash V_2 \rrbracket_{\mathbb{V}} \circ (\mathrm{id} \times \llbracket x_1, \ldots, x_n \vdash V_1 \rrbracket_{\mathbb{V}}) \circ \Delta \\
&\quad \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle \\
&= \llbracket x_1, \ldots x_n \vdash \text{let } x \Leftarrow V_1 \text{ in } V_2 \rrbracket_{\mathbb{V}} \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle
\end{aligned}
$$

where w holds by weakening.

- **Case: abst**

$$
\begin{aligned}
&\llbracket \Gamma \vdash (\lambda x.M)[x_i \mapsto U_i] \rrbracket_{\mathbb{V}} \\
&\quad = \llbracket \Gamma \vdash \lambda x.(M[x_i \mapsto U_i, x \mapsto x]) \rrbracket_{\mathbb{V}} \\
&\quad = \Lambda\Big(\llbracket \Gamma, x : \sigma_1 \vdash M[x_i \mapsto U_i, x \mapsto x] \rrbracket_{\mathbb{C}} \Big) \\
&\quad \stackrel{\mathrm{IH}}{=} \Lambda\Big(\llbracket x_1, \ldots, x_n, x \vdash M \rrbracket_{\mathbb{C}} \circ J \langle \llbracket \Gamma, x \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots \llbracket \Gamma, x \vdash U_n \rrbracket_{\mathbb{V}}, \llbracket \Gamma, x \vdash x \rrbracket_{\mathbb{V}} \rangle \Big) \\
&\quad \stackrel{\mathrm{w}}{=} \Lambda\Big(\llbracket x_1, \ldots, x_n, x \vdash M \rrbracket_{\mathbb{C}} \circ J \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}} \circ \pi_1, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \circ \pi_1, \pi_2 \rangle \Big) \\
&\quad = \Lambda\Big(\llbracket x_1, \ldots, x_n, x \vdash M \rrbracket_{\mathbb{C}} \circ ((J \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle) \otimes \mathrm{id}) \Big) \\
&\quad \stackrel{*}{=} \Lambda\Big(\llbracket x_1, \ldots, x_n, x \vdash M \rrbracket_{\mathbb{C}} \Big) \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle \\
&\quad = (\llbracket x_1, \ldots, x_n \vdash \lambda x.M \rrbracket_{\mathbb{V}}) \circ \langle \llbracket \Gamma \vdash U_1 \rrbracket_{\mathbb{V}}, \ldots, \llbracket \Gamma \vdash U_n \rrbracket_{\mathbb{V}} \rangle
\end{aligned}
$$

where w holds by weakening and $*$ holds by Lemma 2.

- **Case: val-to-comp**

$$
\llbracket \Gamma \vdash V[x_i \mapsto U_i] \rrbracket_{\mathbb{C}}
$$

$$
\begin{aligned}
&= J[\![\Gamma \vdash V[x_i \mapsto U_i]]\!]_{\mathbb{V}} \\
&\overset{\text{IH}}{=} J([\![x_1, \ldots, x_n \vdash V]\!]_{\mathbb{V}} \circ \langle [\![U_1]\!]_{\mathbb{V}}, \ldots, [\![U_n]\!]_{\mathbb{V}} \rangle) \\
&= J([\![x_1, \ldots, x_n \vdash V]\!]_{\mathbb{V}}) \circ J(\langle [\![U_1]\!]_{\mathbb{V}}, \ldots, [\![U_n]\!]_{\mathbb{V}} \rangle)
\end{aligned}
$$

- **Case: efop**

$$
\begin{aligned}
&[\![\Gamma \vdash c[x_i \mapsto U_i] : \tau]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma \vdash c : \tau]\!]_{\mathbb{C}} \\
&\quad = [\![\ \vdash c : \tau]\!]_{\mathbb{C}} \circ J! \\
&\quad = [\![\ \vdash c : \tau]\!]_{\mathbb{C}} \circ J! \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash c : \tau]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}
$$

- **Case: comp-proj**

$$
\begin{aligned}
&[\![\Gamma \vdash (\pi_j M)[x_i \mapsto U_i]]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma \vdash \pi_j(M[x_i \mapsto U_i]]\!]_{\mathbb{C}}) \\
&\quad = J\pi_j \circ [\![\Gamma \vdash (M[x_i \mapsto U_i])]\!]_{\mathbb{C}} \\
&\quad \overset{\text{IH}}{=} J\pi_j \circ [\![x_1, \ldots, x_n \vdash M]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![x_1, \ldots, x_n \vdash \pi_j M]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle
\end{aligned}
$$

- **Case: comp-pair**

$$
\begin{aligned}
&[\![\Gamma \vdash \langle M_1, M_2 \rangle [x_i \mapsto U_i]]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma \vdash \langle M_1[x_i \mapsto U_i], M_2[x_i \mapsto U_i] \rangle]\!]_{\mathbb{C}} \\
&\quad = (\mathrm{id} \otimes [\![\Gamma \vdash M_2[x_i \mapsto U_i]]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1[x_i \mapsto U_i]]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta
\end{aligned}
$$

$$\overset{\text{IH}}{=} (\text{id} \otimes ([\![x_1, \ldots, x_n \vdash M_2]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle))$$

$$\circ (([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle) \otimes \text{id}) \circ J\Delta$$

$$= (\text{id} \otimes ([\![x_1, \ldots, x_n \vdash M_2]\!]_{\mathbb{C}})) \circ (\text{id} \otimes J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle)$$

$$\circ ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \otimes \text{id}) \circ (J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \otimes \text{id})$$

$$\circ J\Delta$$

$$= (\text{id} \otimes ([\![x_1, \ldots, x_n \vdash M_2]\!]_{\mathbb{C}})) \circ ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \otimes \text{id}) \circ J\Delta$$

$$\circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle$$

$$= [\![x_1, \ldots, x_n \vdash \langle M_1, M_2 \rangle]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle$$

- **Case: comp-let**

$$[\![\Gamma \vdash (\text{let } x \Leftarrow M_1 \text{ in } M_2)[x_i \mapsto U_i]]\!]_{\mathbb{C}}$$

$$= [\![\Gamma \vdash \text{let } x \Leftarrow (M_1[x_i \mapsto U_i]) \text{ in } (M_2[x_i \mapsto U_i, x \mapsto x])]\!]_{\mathbb{C}}$$

$$= [\![\Gamma, x \vdash M_2[x_i \mapsto U_i, x \mapsto x]]\!]_{\mathbb{C}} \circ (\text{id} \otimes [\![\Gamma \vdash M_1[x_i \mapsto U_i]]\!]_{\mathbb{C}}) \circ J\Delta$$

$$\overset{\text{IH}}{=} ([\![x_1, \ldots, x_n, x \vdash M_2]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma, x \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma, x \vdash U_n]\!]_{\mathbb{V}}, [\![\Gamma, x \vdash x]\!]_{\mathbb{V}} \rangle)$$

$$\circ (\text{id} \otimes ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle)) \circ J\Delta$$

$$\overset{\text{w}}{=} ([\![x_1, \ldots, x_n, x \vdash M_2]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}} \circ \pi_1, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \circ \pi_1, \pi_2 \rangle))$$

$$\circ (\text{id} \otimes ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle)) \circ J\Delta$$

$$= [\![x_1, \ldots, x_n, x \vdash M_2]\!]_{\mathbb{C}} \circ (J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \otimes \text{id})$$

$$\circ (\text{id} \otimes ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle)) \circ J\Delta$$

$$= [\![x_1, \ldots, x_n, x \vdash M_2]\!]_{\mathbb{C}} \circ (J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle \otimes \text{id})$$

$$\circ (\text{id} \otimes [\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}}) \circ (\text{id} \otimes J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}} \rangle)$$

$$\circ J\Delta$$

$$= [\![x_1, \ldots, x_n, x \vdash M_2]\!]_{\mathbb{C}} \circ (\text{id} \otimes [\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}})$$

$$
\begin{aligned}
&\circ (\mathrm{id} \otimes J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle) \\
&\circ (J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle \otimes \mathrm{id}) \circ J\Delta \\
= {} & [\![x_1, \ldots, x_n, x \vdash M_2]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle \\
= {} & [\![x_1, \ldots, x_n \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle
\end{aligned}
$$

where w holds by weakening.

- **Case: app**

$$
\begin{aligned}
& [\![\Gamma \vdash (M_1 M_2)[x_i \mapsto U_i]]\!]_{\mathbb{C}} \\
& = [\![\Gamma \vdash (M_1[x_i \mapsto U_i])(M_2[x_i \mapsto U_i])]\!]_{\mathbb{C}} \\
& = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2[x_i \mapsto U_i]]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1[x_i \mapsto U_i]]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \\
& \stackrel{\mathrm{IH}}{=} \mathrm{eval} \circ (\mathrm{id} \otimes ([\![x_1, \ldots, x_n \vdash M_2]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle)) \\
& \qquad \circ (([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{C}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{C}}\rangle) \otimes \mathrm{id}) \circ J\Delta \\
& = \mathrm{eval} \circ (\mathrm{id} \otimes ([\![x_1, \ldots, x_n \vdash M_2]\!]_{\mathbb{C}})) \\
& \qquad \circ (\mathrm{id} \otimes J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle) \\
& \qquad \circ ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \\
& \qquad \circ (J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle \otimes \mathrm{id}) \circ J\Delta \\
& = \mathrm{eval} \circ (\mathrm{id} \otimes ([\![x_1, \ldots, x_n \vdash M_2]\!]_{\mathbb{C}})) \circ ([\![x_1, \ldots, x_n \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \\
& \qquad \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle \\
& = [\![x_1, \ldots, x_n \vdash M_1 M_2]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash U_1]\!]_{\mathbb{V}}, \ldots, [\![\Gamma \vdash U_n]\!]_{\mathbb{V}}\rangle
\end{aligned}
$$

□

Theorem 11 (Soundness). *The interpretation of λ_{C} is sound with respect to the* *equational theory described in Figure 4.2.*

That is, if we interpret λ_C *in the closed Freyd-category* $\mathbb{V} \xrightarrow{J} \mathbb{C}$*, then for values* V_1, V_2*, we have* $(\Gamma \vdash V_1 \equiv V_2 : \tau) \implies [\![\Gamma \vdash V_1 : \tau]\!]_{\mathbb{V}} = [\![\Gamma \vdash V_2 : \tau]\!]_{\mathbb{V}}$*, and for terms* M_1, M_2*, we have* $(\Gamma \vdash M_1 \equiv M_2 : \tau) \implies [\![\Gamma \vdash M_1 : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_2 : \tau]\!]_{\mathbb{C}}$*.*

Proof. We are going to prove that each of the rules from Figure 4.2, and reflexivity are sound. We can then prove that symmetry, transitivity, and the congruence rules are sound by induction on the derivation of the $\equiv$-relation.

For the reflexivity, symmetry, transitivity and congruence rules, we are only going to show soundness for $[\![-]\!]_{\mathbb{C}}$, as the proof follows exactly the same way for $[\![-]\!]_{\mathbb{V}}$.

Reflexivity is sound because for any term $(\Gamma \vdash M : \tau)$, $[\![\Gamma \vdash M : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M : \tau]\!]_{\mathbb{C}}$.

The symmetry rule

$$\frac{\Gamma \vdash M_1 \equiv M_2 : \tau}{\Gamma \vdash M_2 \equiv M_1 : \tau}$$

is sound as if we deduce $\Gamma \vdash M_2 \equiv M_1 : \tau$ with this rule, we can apply the inductive hypothesis to the condition of the rule to get $[\![\Gamma \vdash M_1 : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_2 : \tau]\!]_{\mathbb{C}}$, so using the symmetry of equality, $[\![\Gamma \vdash M_2 : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_1 : \tau]\!]_{\mathbb{C}}$.

Similarly, the transitivity rule

$$\frac{\Gamma \vdash M_1 \equiv M_2 : \tau \quad \Gamma \vdash M_2 \equiv M_3 : \tau}{\Gamma \vdash M_1 \equiv M_3 : \tau}$$

is sound because if we deduce $\Gamma \vdash M_1 \equiv M_3 : \tau$ with this rule, we can apply the inductive hypothesis to the condition of the rule to get $[\![\Gamma \vdash M_1 : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_2 : \tau]\!]_{\mathbb{C}}$ and $[\![\Gamma \vdash M_2 : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_3 : \tau]\!]_{\mathbb{C}}$, so using the transitivity of equality, $[\![\Gamma \vdash M_1 : \tau]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_3 : \tau]\!]_{\mathbb{C}}$.

The congruence rules are sound by induction as the interpretation is defined compositionally. E.g., consider the following congruence rule.

$$\frac{\Gamma \vdash M_1 \equiv M_1' : \tau_1 \quad \Gamma \vdash M_2 \equiv M_2' : \tau_2}{\Gamma \vdash \langle M_1, M_2 \rangle \equiv \langle M_1', M_2' \rangle : \tau_1 \times \tau_2}\ .$$

If we deduce $\Gamma \vdash \langle M_1, M_2 \rangle \equiv \langle M_1', M_2' \rangle : \tau_1 \times \tau_2$ with this rule, then we can apply the inductive hypothesis to the condition of the rule to get $[\![\Gamma \vdash M_1 : \tau_1]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_1' : \tau_1]\!]_{\mathbb{C}}$ and $[\![\Gamma \vdash M_2 : \tau_2]\!]_{\mathbb{C}} = [\![\Gamma \vdash M_2' : \tau_2]\!]_{\mathbb{C}}$, so using that the denotations are defined in terms of the denotations of the subterms, $[\![\Gamma \vdash \langle M_1, M_2 \rangle : \tau_1 \times \tau_2]\!]_{\mathbb{C}} = [\![\Gamma \vdash \langle M_1', M_2' \rangle : \tau_1 \times \tau_2]\!]_{\mathbb{C}}$.

So it remains to prove that the rules given explicitly are sound.

- **Case: let**β $\quad$ let $x \Leftarrow V$ in $M \equiv M[x \mapsto V]$

$$\begin{aligned}
&[\![\Gamma \vdash \text{let } x \Leftarrow V \text{ in } M]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma, x \vdash M]\!]_{\mathbb{C}} \circ (\text{id} \otimes [\![\Gamma \vdash V]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = [\![\Gamma, x \vdash M]\!]_{\mathbb{C}} \circ (\text{id} \otimes (J[\![\Gamma \vdash V]\!]_{\mathbb{V}})) \circ J\Delta \\
&\quad = [\![\Gamma, x \vdash M]\!]_{\mathbb{C}} \circ J\langle \text{id}, [\![\Gamma \vdash V]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![\Gamma, x \vdash M]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash \Gamma]\!]_{\mathbb{V}}, [\![\Gamma \vdash V]\!]_{\mathbb{V}} \rangle \\
&\quad \overset{\text{s}}{=} [\![\Gamma \vdash M[\Gamma \mapsto \Gamma, x \mapsto V]]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma \vdash M[x \mapsto V]]\!]_{\mathbb{C}}
\end{aligned}$$

- **Case: prod**β $\quad$ $\pi_i \langle V_1, V_2 \rangle \equiv V_i$

$$\begin{aligned}
&[\![\Gamma \vdash \pi_i \langle V_1, V_2 \rangle]\!]_{\mathbb{V}} \\
&\quad = \pi_i \circ [\![\Gamma \vdash \langle V_1, V_2 \rangle]\!]_{\mathbb{V}} \\
&\quad = \pi_i \circ \langle [\![\Gamma \vdash V_1]\!]_{\mathbb{V}}, [\![\Gamma \vdash V_2]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![\Gamma \vdash V_i]\!]_{\mathbb{V}}
\end{aligned}$$

- **Case: fn**$\boldsymbol{\beta}$ $\quad (\lambda x.M)V \equiv M[x \mapsto V]$

$$
\begin{aligned}
&[\![\Gamma \vdash (\lambda x.M)V]\!]_{\mathbb{C}} \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash V]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash \lambda x.M]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes J[\![\Gamma \vdash V]\!]_{\mathbb{V}}) \circ (J\Lambda([\![\Gamma, x \vdash M]\!]_{\mathbb{C}}) \otimes \mathrm{id}) \circ J\Delta \\
&\quad = (\mathrm{eval} \circ (J\Lambda([\![\Gamma, x \vdash M]\!]_{\mathbb{C}}) \otimes \mathrm{id})) \circ (\mathrm{id} \otimes J[\![\Gamma \vdash V]\!]_{\mathbb{V}}) \circ J\Delta \\
&\quad \stackrel{\beta}{=} [\![\Gamma, x \vdash M]\!]_{\mathbb{C}} \circ J\langle \mathrm{id}, [\![\Gamma \vdash V]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![\Gamma, x \vdash M]\!]_{\mathbb{C}} \circ J\langle [\![\Gamma \vdash \Gamma]\!]_{\mathbb{V}}, [\![\Gamma \vdash V]\!]_{\mathbb{V}} \rangle \\
&\quad \stackrel{\mathrm{s}}{=} [\![\Gamma \vdash M[\Gamma \mapsto \Gamma, x \mapsto V]]\!]_{\mathbb{C}} = [\![\Gamma \vdash M[x \mapsto V]]\!]_{\mathbb{C}}
\end{aligned}
$$

- **Case: let**$\boldsymbol{\eta}$ $\quad \mathrm{let}\ x \Leftarrow M\ \mathrm{in}\ x \equiv M$

$$
\begin{aligned}
&[\![\Gamma \vdash \mathrm{let}\ x \Leftarrow M\ \mathrm{in}\ x]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma, x \vdash x]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\pi_2 \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\pi_2 \circ (J! \otimes \mathrm{id}) \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\pi_2 \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ (J! \otimes \mathrm{id}) \circ J\Delta \\
&\quad \stackrel{*}{=} \rho \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ (J! \otimes \mathrm{id}) \circ J\Delta \\
&\quad \stackrel{\dagger}{=} [\![\Gamma \vdash M]\!]_{\mathbb{C}} \circ \rho \circ (J! \otimes \mathrm{id}) \circ J\Delta \\
&\quad = [\![\Gamma \vdash M]\!]_{\mathbb{C}} \circ J\pi_2 \circ (J! \otimes \mathrm{id}) \circ J\Delta \\
&\quad = [\![\Gamma \vdash M]\!]_{\mathbb{C}}
\end{aligned}
$$

where $*$ holds because $J\pi_2^{I,X} = \rho_X$ because J preserves the premonoidal structure; and $\dagger$ holds because $\rho : I \otimes X \to X$ is natural.

- **Case: prodη** $\langle \pi_1 V, \pi_2 V \rangle \equiv V$

$$
\begin{aligned}
&[\![\Gamma \vdash \langle \pi_1 V, \pi_2 V \rangle]\!]_{\mathbb{V}} \\
&\quad = \langle [\![\Gamma \vdash \pi_1 V]\!]_{\mathbb{V}}, [\![\Gamma \vdash \pi_2 V]\!]_{\mathbb{V}} \rangle \\
&\quad = \langle \pi_1 \circ [\![\Gamma \vdash V]\!]_{\mathbb{V}}, \pi_2 \circ [\![\Gamma \vdash V]\!]_{\mathbb{V}} \rangle \\
&\quad = [\![\Gamma \vdash V]\!]_{\mathbb{V}}
\end{aligned}
$$

- **Case: fnη** $\lambda x.Vx \equiv V$

$$
\begin{aligned}
&[\![\lambda x.Vx]\!]_{\mathbb{V}} \\
&\quad = \Lambda\Big([\![\Gamma, x \vdash Vx]\!]_{\mathbb{C}}\Big) \\
&\quad = \Lambda\Big(\mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash x]\!]_{\mathbb{C}}) \circ ([\![\Gamma, x \vdash V]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta\Big) \\
&\quad = \Lambda\Big(\mathrm{eval} \circ (\mathrm{id} \otimes J\pi_2) \circ (J[\![\Gamma, x \vdash V]\!]_{\mathbb{V}} \otimes \mathrm{id}) \circ J\Delta\Big) \\
&\quad \overset{\mathrm{w}}{=} \Lambda\Big(\mathrm{eval} \circ (\mathrm{id} \otimes J\pi_2) \circ ((J[\![\Gamma \vdash V]\!]_{\mathbb{V}} \circ J\pi_1) \otimes \mathrm{id}) \circ J\Delta\Big) \\
&\quad = \Lambda\Big(\mathrm{eval} \circ ((J[\![\Gamma \vdash V]\!]_{\mathbb{V}}) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes J\pi_2) \circ (J\pi_1 \otimes \mathrm{id}) \circ J\Delta\Big) \\
&\quad = \Lambda\Big(\mathrm{eval} \circ ((J[\![\Gamma \vdash V]\!]_{\mathbb{V}}) \otimes \mathrm{id})\Big) \\
&\quad \overset{\eta}{=} [\![\Gamma \vdash V]\!]_{\mathbb{V}}
\end{aligned}
$$

- **Case: assoc**

  $\text{let } y \Leftarrow (\text{let } x \Leftarrow M_1 \text{ in } M_2) \text{ in } M_3 \equiv \text{let } x \Leftarrow M_1 \text{ in } (\text{let } y \Leftarrow M_2 \text{ in } M_3)$

$$
\begin{aligned}
&[\![\Gamma \vdash \text{let } y \Leftarrow (\text{let } x \Leftarrow M_1 \text{ in } M_2) \text{ in } M_3]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes ([\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta)) \circ J\Delta \\
&\quad = [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}))
\end{aligned}
$$

$$
\begin{aligned}
&\circ (\mathrm{id} \otimes J\Delta) \circ J\Delta \\
&\overset{*}{=} [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}})) \circ a \\
&\quad \circ J(\langle \mathrm{id}, \mathrm{id} \rangle \times \mathrm{id}) \circ J\Delta \\
&\overset{\dagger}{=} [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ a \circ ((\mathrm{id} \otimes \mathrm{id}) \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \\
&\quad \circ (J\langle \mathrm{id}, \mathrm{id} \rangle \otimes \mathrm{id}) \circ J\Delta \\
&= [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ a \circ (J\langle \mathrm{id}, \mathrm{id} \rangle \otimes \mathrm{id}) \\
&\quad \circ ((\mathrm{id} \otimes \mathrm{id}) \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&= [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes J\Delta) \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta
\end{aligned}
$$

where $*$ holds because $(\mathrm{id} \otimes J\Delta) = J(a \circ (\langle \mathrm{id}, \mathrm{id} \rangle \times \mathrm{id}))$ and $\dagger$ holds because a is a natural transformation.

Furthermore,

$$
\begin{aligned}
&[\![\Gamma \vdash \mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ (\mathrm{let}\ y \Leftarrow M_2\ \mathrm{in}\ M_3)]\!]_{\mathbb{C}} = \\
&= [\![\Gamma, x \vdash (\mathrm{let}\ y \Leftarrow M_2\ \mathrm{in}\ M_3)]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&= ([\![\Gamma, x, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta) \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\overset{\mathrm{w}}{=} ([\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ J\langle \pi_1 \circ \pi_1, \pi_2 \rangle \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta) \\
&\quad \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&= [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (J\pi_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&= [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ (J\pi_1 \otimes \mathrm{id}) \circ J\Delta \\
&\quad \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&= [\![\Gamma, y \vdash M_3]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes J\Delta) \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta
\end{aligned}
$$

- **Case: unit** $() \equiv V$

The interpretation of both sides, $[\![\Gamma \vdash () : 1]\!]_{\mathbb{V}}$ and $[\![\Gamma \vdash V : 1]\!]_{\mathbb{V}}$ are morphisms to the terminal object 1, so they have to agree.

- **Case: compproj** $\quad \pi_i M \equiv \text{let } x \Leftarrow M \text{ in } \pi_i x$

$$\begin{aligned}
&[\![\Gamma \vdash \text{let } x \Leftarrow M \text{ in } \pi_i x]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma, x \vdash \pi_i x]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\pi_i \circ [\![\Gamma, x \vdash x]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\pi_i \circ J\pi_2 \circ (\mathrm{id} \otimes [\![\Gamma \vdash M]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\pi_i \circ [\![\Gamma \vdash M]\!]_{\mathbb{C}} \circ J\pi_2 \circ J\Delta \\
&\quad = J\pi_i \circ [\![\Gamma \vdash M]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma \vdash \pi_i M]\!]_{\mathbb{C}}
\end{aligned}$$

- **Case: comppair** $\quad \langle M_1, M_2 \rangle \equiv \text{let } x \Leftarrow M_1 \text{ in } (\text{let } y \Leftarrow M_2 \text{ in } \langle x, y \rangle)$

$$\begin{aligned}
&[\![\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in } (\text{let } y \Leftarrow M_2 \text{ in } \langle x, y \rangle)]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma, x \vdash (\text{let } y \Leftarrow M_2 \text{ in } \langle x, y \rangle)]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = [\![\Gamma, x, y \vdash \langle x, y \rangle]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = J\langle \pi_2 \circ \pi_1, \pi_2 \rangle \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad \overset{\mathrm{w}}{=} J\langle \pi_2 \circ \pi_1, \pi_2 \rangle \circ (\mathrm{id} \otimes ([\![\Gamma \vdash M_2]\!]_{\mathbb{C}} \circ J\pi_1) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = (J\pi_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes J\pi_1) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \\
&\qquad \circ J\Delta \\
&\quad = (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ (J\pi_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes J\pi_1) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \\
&\qquad \circ J\Delta \\
&\quad = (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ J\langle \pi_2, \pi_1 \rangle \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta
\end{aligned}$$

$$
\begin{aligned}
&\overset{*}{=} (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ s \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\overset{\dagger}{=} (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ s \circ J\Delta \\
&= (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\langle \pi_2, \pi_1 \rangle \circ J\Delta \\
&= (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta \\
&= [\![\Gamma \vdash \langle M_1, M_2 \rangle]\!]_{\mathbb{C}}
\end{aligned}
$$

where $*$ holds because J preserves symmetry, and $\dagger$ holds because s is natural.

- **Case: compapp** $\quad M_1 M_2 \equiv \mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ (\mathrm{let}\ y \Leftarrow M_2\ \mathrm{in}\ xy)$

$$
\begin{aligned}
&[\![\Gamma \vdash \mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ (\mathrm{let}\ y \Leftarrow M_2\ \mathrm{in}\ xy)]\!]_{\mathbb{C}} \\
&\quad = [\![\Gamma, x \vdash (\mathrm{let}\ y \Leftarrow M_2\ \mathrm{in}\ xy)]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = [\![\Gamma, x, y \vdash xy]\!]_{\mathbb{C}} \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ J\langle \pi_2 \circ \pi_1, \pi_2 \rangle \circ (\mathrm{id} \otimes [\![\Gamma, x \vdash M_2]\!]_{\mathbb{C}}) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad \overset{\mathrm{w}}{=} \mathrm{eval} \circ J\langle \pi_2 \circ \pi_1, \pi_2 \rangle \circ (\mathrm{id} \otimes ([\![\Gamma \vdash M_2]\!]_{\mathbb{C}} \circ J\pi_1)) \circ J\Delta \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \\
&\qquad\quad \circ J\Delta \\
&\quad = \mathrm{eval} \circ (J\pi_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ (\mathrm{id} \otimes J\pi_1) \circ J\Delta \\
&\qquad\quad \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ (J\pi_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes J\pi_1) \circ J\Delta \\
&\qquad\quad \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ J\langle \pi_2, \pi_1 \rangle \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad \overset{*}{=} \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ s \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_1]\!]_{\mathbb{C}}) \circ J\Delta \\
&\quad \overset{\dagger}{=} \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ s \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\langle \pi_2, \pi_1 \rangle \circ J\Delta \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes [\![\Gamma \vdash M_2]\!]_{\mathbb{C}}) \circ ([\![\Gamma \vdash M_1]\!]_{\mathbb{C}} \otimes \mathrm{id}) \circ J\Delta
\end{aligned}
$$

$$\tau = \beta \mid 1 \mid \tau_1 \times \tau_2 \mid \tau_1 \rightarrow \tau_2$$

Figure 4.4: Objects of $\mathbb{V}$ and $\mathbb{C}$

$$= [\![\Gamma \vdash \langle M_1, M_2 \rangle]\!]_{\mathbb{C}}$$

where $*$ holds because J preserves symmetry, and $\dagger$ holds because s is natural.

This concludes the proof, so the interpretation is indeed sound with respect to the equational theory. □

4.5 Syntactic closed Freyd-category of λ_C

This section describes the syntactic Freyd-category of λ_C with a given signature and proves that it is indeed a closed Freyd-category.

4.5.1 Definition of the syntactic Freyd-category of λ_C

$\mathbb{V} \xrightarrow{J} \mathbb{C}$ where:

Objects of $\mathbb{V}$ and $\mathbb{C}$ are the types of λ_C, as in Figure 4.4.

Morphisms of $\mathbb{V}$ from object τ_1 to τ_2 are equivalence classes of well-typed terms (with a distinguished and fixed free variable x) $x : \tau_1 \vdash V : \tau_2$, and morphisms of $\mathbb{C}$ from object τ_1 to τ_2 are equivalence classes of well-typed terms (with a distinguished and fixed free variable x) $x : \tau_1 \vdash M : \tau_2$ of λ_C, quotiented by the equations from Figure 4.2, as described in Figure 4.5. We describe morphism with contexts for brevity. A term $\Gamma \vdash V : \tau$ or $\Gamma \vdash M : \tau$ for context $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$ corresponds to a morphism from $((\tau_1 \times \tau_2) \ldots) \times \tau_n$ to τ and is essentially a shorthand for $x : ((\tau_1 \times \tau_2) \ldots) \times \tau_n \vdash V[x_1 \mapsto \pi_1 x, \ldots, x_n \mapsto \pi_n x]$ and $x : ((\tau_1 \times \tau_2) \ldots) \times \tau_n \vdash M[x_1 \mapsto \pi_1 x, \ldots, x_n \mapsto \pi_n x]$ respectively. This is consistent with the treatment of contexts in the interpretations of λ_C: in both cases, we simulate n-ary products

morphisms of $\mathbb{V}$	morphisms of $\mathbb{C}$
$\dfrac{}{x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i}$ (var)	$\dfrac{\Gamma \vdash V : \tau}{\Gamma \vdash V : \tau}$ (val-to-comp)
$\dfrac{}{\Gamma \vdash () : 1}$ (unit)	
$\dfrac{\Gamma \vdash V : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i V : \tau_i}$ (val-proj)	$\dfrac{\Gamma \vdash M : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i M : \tau_i}$ (comp-proj)
$\dfrac{\Gamma \vdash V_1 : \tau_1 \quad \Gamma \vdash V_2 : \tau_2}{\Gamma \vdash \langle V_1, V_2 \rangle : \tau_1 \times \tau_2}$ (val-pair)	$\dfrac{\Gamma \vdash M_1 : \tau_1 \quad \Gamma \vdash M_2 : \tau_2}{\Gamma \vdash \langle M_1, M_2 \rangle : \tau_1 \times \tau_2}$ (comp-pair)
$\dfrac{\Gamma \vdash V_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash V_2 : \tau_2}{\Gamma \vdash \text{let } x \Leftarrow V_1 \text{ in } V_2 : \tau_2}$ (val-let)	$\dfrac{\Gamma \vdash M_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash M_2 : \tau_2}{\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2 : \tau_2}$ (comp-let)
$\dfrac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x.M : \tau_1 \to \tau_2}$ (abst)	$\dfrac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1 M_2 : \tau_2}$ (app)
$\dfrac{(c_{prim}, \tau) \in \mathcal{S}_{prim}}{\Gamma \vdash c_{prim} : \tau}$ (prim)	$\dfrac{(c_{efop}, \tau) \in \mathcal{S}_{efop}}{\Gamma \vdash c_{efop} : \tau}$ (efop)
Quotiented by the equations from Figure 4.2	

Figure 4.5: Morphisms of $\mathbb{V}$ and $\mathbb{C}$

with a sequence of binary products associating to the left.

Identity in $\mathbb{V}$ and $\mathbb{C}$ of object τ is the morphism $x : \tau \vdash x : \tau$ which exists in $\mathbb{V}$ by (var) and in $\mathbb{C}$ by (val-to-comp).

Composition of morphisms $x : \tau_1 \vdash M_1 : \tau_2$ and $y : \tau_2 \vdash M_2 : \tau_3$ in $\mathbb{C}$ is

$$(y : \tau_2 \vdash M_2 : \tau_3) \circ (x : \tau_1 \vdash M_1 : \tau_2) = (x : \tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } M_2 : \tau_3).$$

This morphism exists in $\mathbb{C}$ because

$$\dfrac{x : \tau_1 \vdash M_1 : \tau_2 \quad \dfrac{y : \tau_2 \vdash M_2 : \tau_3}{x : \tau_1, y : \tau_2 \vdash M_2 : \tau_3}\text{(weakening)}}{x : \tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } M_2 : \tau_3}\text{(comp-let)}\,.$$

Similarly, using (val-let), define composition in $\mathbb{V}$ as

$$(y : \tau_2 \vdash V_2 : \tau_3) \circ (x : \tau_1 \vdash V_1 : \tau_2) = (x : \tau_1 \vdash \text{let } y \Leftarrow V_1 \text{ in } V_2 : \tau_3).$$

Define **the functor** J to be identity-on-objects, and map a morphism $x : \tau_1 \vdash V : \tau_2$ to the morphism $x : \tau_1 \vdash V : \tau_2$ in $\mathbb{C}$, which exists by (val-to-comps).

4.5.2 $\mathbb{V}$ and $\mathbb{C}$ are categories

Lemma 5. *Composition (in both $\mathbb{V}$ and $\mathbb{C}$) is well-defined with respect to the quotient, i.e., if*

$$x : \tau_1 \vdash M_1 \equiv M_2 : \tau_2$$
$$y : \tau_2 \vdash M_3 \equiv M_4 : \tau_3$$

according to the equational theory, then

$$(y : \tau_2 \vdash M_3 : \tau_3) \circ (x : \tau_1 \vdash M_1 : \tau_2) \equiv (y : \tau_2 \vdash M_4 : \tau_3) \circ (x : \tau_1 \vdash M_2 : \tau_2).$$

Proof. Using the definition of composition,

$$(y : \tau_2 \vdash M_3 : \tau_3) \circ (x : \tau_1 \vdash M_1 : \tau_2) = x : \tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } M_3 : \tau_3$$

and

$$(y : \tau_2 \vdash M_4 : \tau_3) \circ (x : \tau_1 \vdash M_2 : \tau_2) = x : \tau_1 \vdash \text{let } y \Leftarrow M_2 \text{ in } M_4 : \tau_3.$$

These two morphisms agree because the $\equiv$-relation is a congruence. □

Lemma 6. *The identity as defined above satisfies the left- and right-unit rules.*

Proof. Let us prove this for $\mathbb{C}$, as the proof follows exactly analogously for $\mathbb{V}$.

$$\begin{aligned}
&\mathrm{id}_{\tau_2} \circ (x : \tau_1 \vdash M : \tau_2) \\
&\quad = (y : \tau_2 \vdash y : \tau_2) \circ (x : \tau_1 \vdash M : \tau_2) \\
&\quad = x : \tau_1 \vdash \text{let } y \Leftarrow M \text{ in } y : \tau_2 \\
&\quad \overset{1\eta}{=} x : \tau_1 \vdash M : \tau_2
\end{aligned}$$

$$\begin{aligned}
&(x : \tau_1 \vdash M : \tau_2) \circ \mathrm{id}_{\tau_1} \\
&\quad = (x : \tau_1 \vdash M : \tau_2) \circ (y : \tau_1 \vdash y : \tau_1) \\
&\quad = y : \tau_1 \vdash \text{let } x \Leftarrow y \text{ in } M : \tau_2 \\
&\quad \overset{1\beta}{=} y : \tau_1 \vdash M[x \mapsto y] : \tau_2 \\
&\quad = x : \tau_1 \vdash M : \tau_2
\end{aligned}$$

□

Lemma 7. *Composition as defined above is associative.*

Proof. Let us prove this for composition in $\mathbb{C}$, as the proof follows exactly analogously for $\mathbb{V}$.

$$\begin{aligned}
&((z : \tau_3 \vdash M_3 : \tau_4) \circ (y : \tau_2 \vdash M_2 : \tau_3)) \circ (x : \tau_1 \vdash M_1 : \tau_2) \\
&\quad = (y : \tau_2 \vdash \text{let } z \Leftarrow M_2 \text{ in } M_3 : \tau_4) \circ (x : \tau_1 \vdash M_1 : \tau_2) \\
&\quad = (x : \tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } (\text{let } z \Leftarrow M_2 \text{ in } M_3) : \tau_4) \\
&\quad \overset{\text{a}}{=} (x : \tau_1 \vdash \text{let } z \Leftarrow (\text{let } y \Leftarrow M_1 \text{ in } M_2) \text{ in } M_3 : \tau_4) \\
&\quad = (z : \tau_3 \vdash M_3 : \tau_4) \circ (x : \tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } M_2 : \tau_3)
\end{aligned}$$

$$= (z : \tau_3 \vdash M_3 : \tau_4) \circ ((y : \tau_2 \vdash M_2 : \tau_3) \circ (x : \tau_1 \vdash M_1 : \tau_2))$$

□

Combining these lemmas, we can deduce the following proposition.

Proposition 1. $\mathbb{V}$ *and* $\mathbb{C}$ *are categories.*

4.5.3 J is an identity-on-objects functor

Note that it is identity-on-objects by definition, and it maps a morphism between objects τ_1 and τ_2 in $\mathbb{V}$ to a morphism between τ_1 and τ_2 in $\mathbb{C}$ as required.

Lemma 8. *J is well-defined with respect to the quotient, i.e., if*

$$x : \tau_1 \vdash M_1 \equiv M_2 : \tau_2,$$

then $J(x : \tau_1 \vdash M_1 : \tau_2) \equiv J(x : \tau_1 \vdash M_2 : \tau_2)$.

Proof. We quotient the morphisms by the same set of rules, so if $(x : \tau_1 \vdash M_1 : \tau_2) \equiv (x : \tau_1 \vdash M_2 : \tau_2)$ in $\mathbb{V}$, then $(x : \tau_1 \vdash M_1 : \tau_2) \equiv (x : \tau_1 \vdash M_2 : \tau_2)$ in $\mathbb{C}$, i.e., $J(x : \tau_1 \vdash M_1 : \tau_2) \equiv J(x : \tau_1 \vdash M_2 : \tau_2)$. □

Lemma 9. *J respects identity morphisms.*

Proof.

$$J(\mathrm{id}^{\mathbb{V}}_\tau) = J(x : \tau \vdash x : \tau) = (x : \tau \vdash x : \tau) = \mathrm{id}^{\mathbb{C}}_\tau$$

□

Lemma 10. *J respects composition.*

Proof.

$$\begin{aligned}
&J((y:\tau_2 \vdash V_2:\tau_3) \circ_{\mathbb{V}} (x:\tau_1 \vdash V_1:\tau_2)) \\
&\quad = J(x:\tau_1 \vdash \text{let } y \Leftarrow V_1 \text{ in } V_2:\tau_3) \\
&\quad = x:\tau_1 \vdash \text{let } y \Leftarrow V_1 \text{ in } V_2:\tau_3 \\
&\quad = (y:\tau_2 \vdash V_2:\tau_3) \circ_{\mathbb{C}} (x:\tau_1 \vdash V_1:\tau_2) \\
&\quad = J(y:\tau_2 \vdash V_2:\tau_3) \circ_{\mathbb{C}} J(x:\tau_1 \vdash V_1:\tau_2)
\end{aligned}$$

□

Combining these lemmas, we can deduce the following proposition.

Proposition 2. *J is an identity-on-objects functor.*

4.5.4 $\mathbb{V}$ has finite products

Lemma 11. *The object corresponding to type 1 is a* ***terminal object*** *in $\mathbb{V}$.*

Proof. For any object τ in $\mathbb{V}$, there is a unique morphism $\tau \to 1$ in $\mathbb{V}$, $(x:\tau \vdash ():1)$, which exists by the (unit) rule, and it is unique because every $(x:\tau \vdash V:1)$ morphism is equivalent to it by the (unit) equivalence rule. □

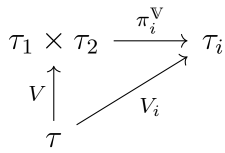
Lemma 12. *$(\tau_1 \times \tau_2, \pi_1^{\mathbb{V}}, \pi_2^{\mathbb{V}})$ for $\pi_i^{\mathbb{V}}:\tau_1 \times \tau_2 \to \tau_i$ given by*

$$\pi_i^{\mathbb{V}} = (x:\tau_1 \times \tau_2 \vdash \pi_i x:\tau_i)$$

is a ***binary product*** *for objects τ_1, τ_2 in $\mathbb{V}$.*

Proof. We are required to prove that this has the universal property, i.e., for any object τ and morphisms $x:\tau \vdash V_i:\tau_i$, there is a unique morphism $x:\tau \vdash V:\tau_1 \times \tau_2$

such that



commutes.

Indeed, we can define $V = \langle V_1, V_2 \rangle$ which is a morphism in $\mathbb{V}$ between the appropriate objects by rule (val-pair), and it has the required property:

$$
\begin{aligned}
&(y : \tau_1 \times \tau_2 \vdash \pi_i y : \tau_i) \circ (x : \tau \vdash \langle V_1, V_2 \rangle : \tau_1 \times \tau_2) \\
&= x : \tau \vdash \text{let } y \Leftarrow \langle V_1, V_2 \rangle \text{ in } \pi_i y : \tau_i \\
&\overset{1\beta}{=} x : \tau \vdash \pi_i \langle V_1, V_2 \rangle : \tau_i \\
&\overset{\text{p}\beta}{=} x : \tau \vdash V_i : \tau_i
\end{aligned}
$$

Furthermore, any V with the above property is $\equiv$-equal to $\langle V_1, V_2 \rangle$, because:

$$
\begin{aligned}
&x : \tau \vdash V : \tau_1 \times \tau_2 \\
&\overset{\text{p}\eta}{=} x : \tau \vdash \langle \pi_1 V, \pi_2 V \rangle : \tau_1 \times \tau_2 \\
&= x : \tau \vdash \langle (\pi_1 y_1)[y_1 \mapsto V], (\pi_2 y_2)[y_2 \mapsto V] \rangle : \tau_1 \times \tau_2 \\
&= x : \tau \vdash \langle \text{let } y_1 \Leftarrow V \text{ in } \pi_1 y_1, \text{let } y_2 \Leftarrow V \text{ in } \pi_2 y_2 \rangle : \tau_1 \times \tau_2 \\
&= x : \tau \vdash \langle \pi_1^{\mathbb{V}} \circ V, \pi_2^{\mathbb{V}} \circ V \rangle : \tau_1 \times \tau_2 \\
&= x : \tau \vdash \langle V_1, V_2 \rangle : \tau_1 \times \tau_2
\end{aligned}
$$

□

So $\mathbb{V}$ has a terminal object and binary products, so we can deduce the following proposition.

Proposition 3. $\mathbb{V}$ *is a category with finite products.*

Corollary 1. $\mathbb{V}$ *is a premonoidal category with the premonoidal structure given by:*

$$\tau \rtimes_{\mathbb{V}} (x : \tau_1 \vdash V : \tau_2) = \tau \times_{\mathbb{V}} (x : \tau_1 \vdash V : \tau_2) = (y : \tau \times \tau_1 \vdash \langle \pi_1 y, M[x \mapsto \pi_2 y] \rangle)$$
$$(x : \tau_1 \vdash V : \tau_2) \ltimes_{\mathbb{V}} \tau = (x : \tau_1 \vdash V : \tau_2) \times_{\mathbb{V}} \tau = (y : \tau_1 \times \tau \vdash \langle M[x \mapsto \pi_1 y], \pi_2 y \rangle)$$

$$\begin{aligned} I &= 1 \\ a^{\mathbb{V}}_{\tau_1,\tau_2,\tau_3} &= x : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 x), \langle \pi_2(\pi_1 x), \pi_2 x \rangle \rangle \\ \lambda^{\mathbb{V}}_{\tau} &= (x : \tau \times I \vdash \pi_1 x) \\ \rho^{\mathbb{V}}_{\tau} &= (x : I \times \tau \vdash \pi_2 x) \end{aligned}$$

Proof. $\mathbb{V}$ has finite products, so it is a monoidal category, e.g., as described in [15, Chapter VII], hence it is also a premonoidal category. The structure is derived from the finite products of $\mathbb{V}$. □

4.5.5 $\mathbb{C}$ is a premonoidal category

Description of the premonoidal structure

Let us define the premonoidal structure on $\mathbb{C}$ as follows.

For objects τ_1, τ_2, define $\tau_1 \otimes \tau_2 := \tau_1 \times \tau_2$.

For a particular objects τ, define the functors $\tau \rtimes -$ and $- \ltimes \tau$ as follows:

$$\tau \rtimes \tau' := \tau \otimes \tau' = \tau \times \tau'$$
$$\tau \rtimes (x : \tau_1 \vdash M : \tau_2) := (y : \tau \times \tau_1 \vdash \langle \pi_1 y, M[x \mapsto \pi_2 y] \rangle : \tau \times \tau_2)$$
$$\tau' \ltimes \tau := \tau' \otimes \tau = \tau' \times \tau$$
$$(x : \tau_1 \vdash M : \tau_2) \ltimes \tau := (y : \tau_1 \times \tau \vdash \langle M[x \mapsto \pi_1 y], \pi_2 y \rangle : \tau_2 \times \tau)$$

And define the corresponding premonoidal structure as:

$$
\begin{aligned}
I &:= 1 \\
a_{\tau_1,\tau_2,\tau_3} &:= (x : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 x), \langle \pi_2(\pi_1 x), \pi_2 x \rangle \rangle : \tau_1 \times (\tau_2 \times \tau_3)) \\
\lambda_\tau &:= (x : \tau \times I \vdash \pi_1 x : \tau) \\
\rho_\tau &:= (x : I \times \tau \vdash \pi_2 x : \tau)
\end{aligned}
$$

Lemma 13. *$\tau \rtimes -$ and $- \ltimes \tau$ are indeed functors.*

Proof. $\tau \rtimes -$ respects identities:

$$
\begin{aligned}
&\tau \rtimes \mathrm{id}'_\tau \\
&\quad = (y : \tau \times \tau' \vdash \langle \pi_1 y, \pi_2 y \rangle : \tau \times \tau') \\
&\quad \overset{\mathrm{p}\eta}{=} (y : \tau \times \tau' \vdash y : \tau \times \tau') \\
&\quad = \mathrm{id}_{\tau \times \tau'}
\end{aligned}
$$

And it respects composition:

$$
\begin{aligned}
&(\tau \rtimes (y : \tau_2 \vdash M_2 : \tau_3)) \circ (\tau \rtimes (x : \tau_1 \vdash M_1 : \tau_2)) \\
&\quad = (z_2 : \tau \times \tau_2 \vdash \langle \pi_1 z_2, M_2[y \to \pi_2 z_2] \rangle) \circ (z_1 : \tau \times \tau_1 \vdash \langle \pi_1 z_1, M_1[x \to \pi_2 z_1] \rangle) \\
&\quad = z_1 : \tau \times \tau_1 \vdash \text{let } z_2 \Leftarrow (\langle \pi_1 z_1, M_1[x \to \pi_2 z_1] \rangle) \text{ in } \langle \pi_1 z_2, M_2[y \to \pi_2 z_2] \rangle \\
&\quad \overset{\mathrm{cp}}{=} z_1 : \tau \times \tau_1 \vdash \\
&\qquad \text{let } z_2 \Leftarrow (\text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } \langle \pi_1 z_1, z_3 \rangle) \text{ in } \langle \pi_1 z_2, M_2[y \to \pi_2 z_2] \rangle \\
&\quad \overset{\mathrm{a}}{=} z_1 : \tau \times \tau_1 \vdash \\
&\qquad \text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } (\text{let } z_2 \Leftarrow \langle \pi_1 z_1, z_3 \rangle \text{ in } \langle \pi_1 z_2, M_2[y \to \pi_2 z_2] \rangle)
\end{aligned}
$$

$$
\begin{aligned}
&\stackrel{\mathrm{l}\beta}{=} z_1 : \tau \times \tau_1 \vdash \text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } \langle \pi_1 \langle \pi_1 z_1, z_3 \rangle, M_2[y \to \pi_2 \langle \pi_1 z_1, z_3 \rangle] \rangle \\
&\stackrel{\mathrm{p}\beta}{=} z_1 : \tau \times \tau_1 \vdash \text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } \langle \pi_1 z_1, M_2[y \to z_3] \rangle \\
&\stackrel{\mathrm{cp}}{=} z_1 : \tau \times \tau_1 \vdash \text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } (\text{let } z_4 \Leftarrow M_2[y \to z_3] \text{ in } \langle \pi_1 z_1, z_4 \rangle) \\
&\stackrel{\mathrm{a}}{=} z_1 : \tau \times \tau_1 \vdash \text{let } z_4 \Leftarrow (\text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } M_2[y \to z_3]) \text{ in } \langle \pi_1 z_1, z_4 \rangle \\
&\stackrel{\mathrm{cp}}{=} z_1 : \tau \times \tau_1 \vdash \langle \pi_1 z_1, (\text{let } z_3 \Leftarrow M_1[x \to \pi_2 z_1] \text{ in } M_2[y \to z_3]) \rangle \\
&= z : \tau \times \tau_1 \vdash \langle \pi_1 z, (\text{let } y \Leftarrow M_1[x \mapsto \pi_2 z] \text{ in } M_2) \rangle : \tau \times \tau_3 \\
&= z : \tau \times \tau_1 \vdash \langle \pi_1 z, (\text{let } y \Leftarrow M_1 \text{ in } M_2)[x \mapsto \pi_2 z] \rangle : \tau \times \tau_3 \\
&= \tau \rtimes (x : \tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } M_2 : \tau_3) \\
&= \tau \rtimes ((y : \tau_2 \vdash M_2 : \tau_3) \circ (x : \tau_1 \vdash M_1 : \tau_2))
\end{aligned}
$$

Hence it is a functor. Similarly, $- \ltimes \tau$ is also a functor. □

Lemma 14. *For each $(x : \tau_1 \vdash V : \tau_1')$ and $(y : \tau_2 \vdash M : \tau_2')$,*

$$
\begin{array}{ccc}
\tau_1 \times \tau_2 & \xrightarrow{\tau_1 \rtimes (y:\tau_2 \vdash M:\tau_2')} & \tau_1 \times \tau_2' \\
{\scriptstyle (x:\tau_1 \vdash V:\tau_1') \ltimes \tau_2} \downarrow & & \downarrow {\scriptstyle (x:\tau_1 \vdash V:\tau_1') \ltimes \tau_2'} \\
\tau_1' \times \tau_2 & \xrightarrow[\tau_1' \rtimes (y:\tau_2 \vdash M:\tau_2')]{} & \tau_1' \times \tau_2'
\end{array}
$$

and

$$
\begin{array}{ccc}
\tau_2 \times \tau_1 & \xrightarrow{(y:\tau_2 \vdash M:\tau_2') \ltimes \tau_1} & \tau_2' \times \tau_1 \\
{\scriptstyle \tau_2 \rtimes (x:\tau_1 \vdash V:\tau_1')} \downarrow & & \downarrow {\scriptstyle \tau_2' \rtimes (x:\tau_1 \vdash V:\tau_1')} \\
\tau_2 \times \tau_1' & \xrightarrow[(y:\tau_2 \vdash M:\tau_2') \ltimes \tau_1']{} & \tau_2' \times \tau_1'
\end{array}
$$

commute, i.e., $x : \tau_1 \vdash V : \tau_1'$ is central.

Proof. Let us first consider the two paths in the first square.

$$
\begin{aligned}
&((x : \tau_1 \vdash V : \tau_1') \ltimes \tau_2') \circ (\tau_1 \rtimes (y : \tau_2 \vdash M : \tau_2')) \\
&\quad = (y_1 : \tau_1 \times \tau_2' \vdash \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1 \rangle) \circ (y_2 : \tau_1 \times \tau_2 \vdash \langle \pi_1 y_2, M[y \mapsto \pi_2 y_2] \rangle)
\end{aligned}
$$

$$= y_2 : \tau_1 \times \tau_2 \vdash \text{let } y_1 \Leftarrow \langle \pi_1 y_2, M[y \mapsto \pi_2 y_2]\rangle \text{ in } \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle$$

$$\stackrel{\text{cp}}{=} y_2 : \tau_1 \times \tau_2 \vdash$$

$$\text{let } y_1 \Leftarrow (\text{let } z \Leftarrow M[y \mapsto \pi_2 y_2] \text{ in } \langle \pi_1 y_2, z\rangle) \text{ in } \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle$$

$$\stackrel{\text{a}}{=} y_2 : \tau_1 \times \tau_2 \vdash$$

$$\text{let } z \Leftarrow M[y \mapsto \pi_2 y_2] \text{ in } (\text{let } y_1 \Leftarrow \langle \pi_1 y_2, z\rangle \text{ in } \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle)$$

$$\stackrel{\text{l}\beta}{=} y_2 : \tau_1 \times \tau_2 \vdash \text{let } z \Leftarrow M[y \mapsto \pi_2 y_2] \text{ in } (\langle V[x \mapsto \pi_1\langle \pi_1 y_2, z\rangle], \pi_2\langle \pi_1 y_2, z\rangle\rangle)$$

$$\stackrel{\text{p}\beta}{=} y_2 : \tau_1 \times \tau_2 \vdash \text{let } z \Leftarrow M[y \mapsto \pi_2 y_2] \text{ in } \langle V[x \mapsto \pi_1 y_2], z\rangle$$

and

$$(\tau_1' \rtimes (y : \tau_2 \vdash M : \tau_2')) \circ ((x : \tau_1 \vdash V : \tau_1') \ltimes \tau_2)$$

$$= (y_2 : \tau_1' \times \tau_2 \vdash \langle \pi_1 y_2, M[y \mapsto \pi_2 y_2]\rangle) \circ (y_1 : \tau_1 \times \tau_2 \vdash \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle)$$

$$\stackrel{\text{cp}}{=} (y_2 : \tau_1' \times \tau_2 \vdash \text{let } z \Leftarrow M[y \mapsto \pi_2 y_2] \text{ in } \langle \pi_1 y_2, z\rangle)$$

$$\circ (y_1 : \tau_1 \times \tau_2 \vdash \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle)$$

$$= y_1 : \tau_1 \times \tau_2 \vdash$$

$$\text{let } y_1 \Leftarrow \langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle \text{ in } (\text{let } z \Leftarrow M[y \mapsto \pi_2 y_2] \text{ in } \langle \pi_1 y_2, z\rangle)$$

$$\stackrel{\text{l}\beta}{=} y_1 : \tau_1 \times \tau_2 \vdash$$

$$\text{let } z \Leftarrow M[y \mapsto \pi_2\langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle] \text{ in } \langle \pi_1\langle V[x \mapsto \pi_1 y_1], \pi_2 y_1\rangle, z\rangle$$

$$\stackrel{\text{p}\beta}{=} y_1 : \tau_1 \times \tau_2 \vdash \text{let } z \Leftarrow M[y \mapsto \pi_2 y_1] \text{ in } \langle V[x \mapsto \pi_1 y_1], z\rangle$$

So the two paths agree, so the first square indeed commutes.

Similarly but in the proof, the π_1 and π_2 and the positions in $\langle -, - \rangle$ swapped, the second square commutes as well. □

Hence all value morphisms are central. Note that these are the morphisms of $\mathbb{C}$ that are of the form Jf for a morphism f of $\mathbb{V}$.

Note in particular, that a_{τ_1,τ_2,τ_3}, λ_τ, ρ_τ are values, so they are central.

Lemma 15. *J strictly preserves premonoidal structure, i.e., for any objects τ_1, τ_2, τ_3, τ and morphism $x : \tau_1 \vdash V : \tau_2$ in $\mathbb{V}$,*

$$a_{\tau_1,\tau_2,\tau_3} = Ja^{\mathbb{V}}_{\tau_1,\tau_2,\tau_3}$$
$$\lambda_\tau = J\lambda^{\mathbb{V}}_\tau$$
$$\rho_\tau = J\rho^{\mathbb{V}}_\tau$$

and

$$\tau \rtimes (x : \tau_1 \to V : \tau_2) = J(\tau \rtimes_{\mathbb{V}} (x : \tau_1 \to V : \tau_2))$$
$$(x : \tau_1 \to V : \tau_2) \ltimes \tau = J((x : \tau_1 \to V : \tau_2) \ltimes_{\mathbb{V}} \tau)$$

Proof. This holds trivially by Corollary 1 and the definition of J. □

Lemma 16. *The* ***triangle law*** *and the* ***pentagon law*** *for $\mathbb{C}$ with the claimed premonoidal structure defined above holds.*

Proof. J is a functor, so it preserves commuting diagrams, so using that the triangle law and pentagon law holds for $\mathbb{V}$, it also holds for $\mathbb{C}$. □

Lemma 17. *λ and ρ are natural transformations.*

Proof. To see that λ is natural, required to prove that the following diagram commutes.

$$\begin{array}{ccc} \tau_1 \times I & \xrightarrow{(x:\tau_1 \vdash M:\tau_2) \ltimes I} & \tau_2 \times I \\ {\scriptstyle \lambda_{\tau_1}} \downarrow & & \downarrow {\scriptstyle \lambda_{\tau_2}} \\ \tau_1 & \xrightarrow[(x:\tau_1 \vdash M:\tau_2)]{} & \tau_2 \end{array}$$

$$\begin{aligned}
&\lambda_{\tau_2} \circ ((x : \tau_1 \vdash M : \tau_2) \ltimes I) \\
&\quad = (w : \tau_2 \times I \vdash \pi_1 w : \tau_2) \circ (y : \tau_1 \times I \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } \langle z, \pi_2 y \rangle \\
&\quad = y : \tau_1 \times I \vdash \text{let } w \Leftarrow (\text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } \langle z, \pi_2 y \rangle) \text{ in } \pi_1 w : \tau_2 \\
&\quad \overset{\text{a}}{=} y : \tau_1 \times I \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } (\text{let } w \Leftarrow \langle z, \pi_2 y \rangle \text{ in } \pi_1 w) : \tau_2 \\
&\quad \overset{\text{l}\beta}{=} y : \tau_1 \times I \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } \pi_1 \langle z, \pi_2 y \rangle : \tau_2 \\
&\quad \overset{\text{p}\beta}{=} y : \tau_1 \times I \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } z \\
&\quad \overset{\text{l}\eta}{=} y : \tau_1 \times I \vdash M[x \mapsto \pi_1 y] : \tau_2 \\
&\quad = y : \tau_1 \times I \vdash \text{let } x \Leftarrow \pi_1 y \text{ in } M : \tau_2 \\
&\quad = (x : \tau_1 \vdash M : \tau_2) \circ (y : \tau_1 \times I \vdash \pi_1 y : \tau_1) \\
&\quad = (x : \tau_1 \vdash M : \tau_2) \circ \lambda_{\tau_1}
\end{aligned}$$

Hence λ is indeed natural. Similarly, ρ is also natural. □

Lemma 18. *As defined above, a is a natural transformation with components $a_{\tau_1,\tau_2,\tau_3} : (\tau_1 \otimes \tau_2) \otimes \tau_3 \to \tau_1 \otimes (\tau_2 \otimes \tau_3)$.*

Proof. There are three naturality-squares to consider, one for each of τ_1, τ_2, τ_3. We will consider these in turn, and confirm that the two paths agree in each of them.

$$\begin{array}{ccc}
(\tau_1 \times \tau_2) \times \tau_3 & \xrightarrow{a_{\tau_1,\tau_2,\tau_3}} & \tau_1 \times (\tau_2 \times \tau_3) \\
{\scriptstyle ((x:\tau \vdash M:\tau_1') \ltimes \tau_2) \ltimes \tau_3} \downarrow & & \downarrow {\scriptstyle (x:\tau_1 \vdash M:\tau_1') \ltimes (\tau_2 \times \tau_3)} \\
(\tau_1' \times \tau_2) \times \tau_3 & \xrightarrow[a_{\tau_1',\tau_2,\tau_3}]{} & \tau_1' \times (\tau_2 \times \tau_3)
\end{array}$$

$$\begin{aligned}
&((x : \tau_1 \vdash M : \tau_1') \ltimes (\tau_2 \times \tau_3)) \circ a_{\tau_1,\tau_2,\tau_3} \\
&\quad = (y : \tau_1 \times (\tau_2 \times \tau_3) \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } \langle z, \pi_2 y \rangle : \tau_1' \times (\tau_2 \times \tau_3)) \\
&\qquad \circ (w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 w), \langle \pi_2(\pi_1 w), \pi_2 w \rangle \rangle : \tau_1 \times (\tau_2 \times \tau_3))
\end{aligned}$$

$$= w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } y \Leftarrow \langle \pi_1(\pi_1 w), \langle \pi_2(\pi_1 w), \pi_2 w \rangle\rangle$$
$$\text{in } (\text{let } z \Leftarrow M[x \mapsto \pi_1 y] \text{ in } \langle z, \pi_2 y\rangle) : \tau_1' \times (\tau_2 \times \tau_3)$$
$$\stackrel{1\beta}{=} w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1\langle \pi_1(\pi_1 w), \langle \pi_2(\pi_1 w), \pi_2 w\rangle\rangle]$$
$$\text{in } \langle z, \pi_2\langle \pi_1(\pi_1 w), \langle \pi_2(\pi_1 w), \pi_2 w\rangle\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3)$$
$$\stackrel{\mathrm{p}\beta}{=} w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_1(\pi_1 w)]$$
$$\text{in } \langle z, \langle \pi_2(\pi_1 w), \pi_2 w\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3)$$

$$a_{\tau_1', \tau_2\, \tau_3} \circ (((x : \tau_1 \vdash M : \tau_1') \ltimes \tau_2) \ltimes \tau_3)$$
$$= (q : (\tau_1' \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3))$$
$$\circ\, (w : (\tau_1 \times \tau_2) \times \tau_3 \vdash$$
$$\text{let } z_2 \Leftarrow (\text{let } z_1 \Leftarrow M[x \mapsto \pi_1 w_1] \text{ in } \langle z_1, \pi_2 w_1\rangle)[w_1 \mapsto \pi_1 w]$$
$$\text{in } \langle z_2, \pi_2 w\rangle : (\tau_1' \times \tau_2) \times \tau_3)$$
$$= (q : (\tau_1' \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3))$$
$$\circ\, (w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_2 \Leftarrow (\text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)] \text{ in } \langle z_1, \pi_2(\pi_1 w)\rangle)$$
$$\text{in } \langle z_2, \pi_2 w\rangle : (\tau_1' \times \tau_2) \times \tau_3)$$
$$\stackrel{\mathrm{a}}{=} (q : (\tau_1' \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3))$$
$$\circ\, (w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)]$$
$$\text{in let } z_2 \Leftarrow \langle z_1, \pi_2(\pi_1 w)\rangle \text{ in } \langle z_2, \pi_2 w\rangle : (\tau_1' \times \tau_2) \times \tau_3)$$
$$\stackrel{1\beta}{=} (q : (\tau_1' \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3))$$
$$\circ\, (w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)]$$
$$\text{in let } z_2 \Leftarrow \langle z_1, \pi_2(\pi_1 w)\rangle \text{ in } \langle z_2, \pi_2 w\rangle : (\tau_1' \times \tau_2) \times \tau_3)$$

$$
\begin{aligned}
&= (q : (\tau_1' \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3)) \\
&\qquad \circ (w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)] \\
&\qquad\qquad \text{in } \langle\langle z_1, \pi_2(\pi_1 w)\rangle, \pi_2 w\rangle : (\tau_1' \times \tau_2) \times \tau_3) \\
&= w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } q \Leftarrow (\text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)] \\
&\qquad\qquad \text{in } \langle\langle z_1, \pi_2(\pi_1 w)\rangle, \pi_2 w\rangle) \\
&\qquad\qquad \text{in } \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3) \\
&\overset{\text{a}}{=} w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)] \\
&\qquad\qquad \text{in let } q \Leftarrow \langle\langle z_1, \pi_2(\pi_1 w)\rangle, \pi_2 w\rangle \\
&\qquad\qquad \text{in } \langle \pi_1(\pi_1 q), \langle \pi_2(\pi_1 q), \pi_2 q\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3) \\
&\overset{\text{l}\beta}{=} w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)] \\
&\qquad\qquad \text{in } \langle \pi_1(\pi_1\langle\langle z_1, \pi_2(\pi_1 w)\rangle, \pi_2 w\rangle), \\
&\qquad\qquad\quad \langle \pi_2(\pi_1\langle\langle z_1, \pi_2(\pi_1 w)\rangle, \pi_2 w\rangle), \pi_2\langle\langle z_1, \pi_2(\pi_1 w)\rangle, \pi_2 w\rangle\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3) \\
&= w : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z_1 \Leftarrow M[x \mapsto \pi_1(\pi_1 w)] \\
&\qquad\qquad \text{in } \langle z_1, \langle \pi_2(\pi_1 w), \pi_2 w\rangle\rangle : \tau_1' \times (\tau_2 \times \tau_3)
\end{aligned}
$$

$$
\begin{array}{ccc}
(\tau_1 \times \tau_2) \times \tau_3 & \xrightarrow{a_{\tau_1,\tau_2,\tau_3}} & \tau_1 \times (\tau_2 \times \tau_3) \\
{\scriptstyle (\tau_1 \rtimes x:\tau_2 \vdash M:\tau_2') \ltimes \tau_3}\downarrow & & \downarrow{\scriptstyle \tau_1 \rtimes ((x:\tau_2 \vdash M:\tau_2') \ltimes \tau_3)} \\
(\tau_1 \times \tau_2') \times \tau_3 & \xrightarrow[a_{\tau_1,\tau_2',\tau_3}]{} & \tau_1 \times (\tau_2' \times \tau_3)
\end{array}
$$

$$
\begin{aligned}
&(\tau_1 \rtimes ((x : \tau_2 \vdash M : \tau_2') \ltimes \tau_3)) \circ a_{\tau_1,\tau_2,\tau_3} \\
&\quad = (y_1 : \tau_1 \times (\tau_2 \times \tau_3) \vdash \langle \pi_1 y_1, \langle M[x \mapsto \pi_1(\pi_2 y_1)], \pi_2(\pi_2 y_1)\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)) \\
&\qquad\quad \circ (y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2\rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3)) \\
&\quad = y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } y_1 \Leftarrow \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2\rangle\rangle
\end{aligned}
$$

$$\text{in } \langle \pi_1 y_1, \langle M[x \mapsto \pi_1(\pi_2 y_1)], \pi_2(\pi_2 y_1)\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)$$

$$\stackrel{1\beta}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1 \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2\rangle\rangle,$$

$$\langle M[x \mapsto \pi_1(\pi_2 \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2\rangle\rangle)],$$

$$\pi_2(\pi_2 \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2\rangle\rangle)\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)$$

$$\stackrel{p\beta}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 y_2), \langle M[x \mapsto \pi_2(\pi_1 y_2)], \pi_2 y_2\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)$$

$$a_{\tau_1,\tau_2',\tau_3} \circ (\tau_1 \rtimes ((x : \tau_2 \vdash M : \tau_2') \ltimes \tau_3))$$

$$= (y_1 : (\tau_1 \times \tau_2') \times \tau_3 \vdash \langle \pi_1(\pi_1 y_1), \langle \pi_2(\pi_1 y_1), \pi_2 y_1\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3))$$

$$\circ\, (y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle\langle \pi_1(\pi_1 y_2), M[x \mapsto \pi_2(\pi_1 y_2)]\rangle, \pi_2 y_2\rangle : (\tau_1 \times \tau_2') \times \tau_3)$$

$$= y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash$$

$$\text{let } y_1 \Leftarrow \langle\langle \pi_1(\pi_1 y_2), M[x \mapsto \pi_2(\pi_1 y_2)]\rangle, \pi_2 y_2\rangle$$

$$\text{in } \langle \pi_1(\pi_1 y_1), \langle \pi_2(\pi_1 y_1), \pi_2 y_1\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)$$

$$= y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } y_1 \Leftarrow (\text{let } z \Leftarrow M[x \mapsto \pi_2(\pi_1 y_2)]$$

$$\text{in } \langle\langle \pi_1(\pi_1 y_2), z\rangle, \pi_2 y_2\rangle)$$

$$\text{in } \langle \pi_1(\pi_1 y_1), \langle \pi_2(\pi_1 y_1), \pi_2 y_1\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)$$

$$\stackrel{a}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2(\pi_1 y_2)]$$

$$\text{in } (\text{let } y_1 \Leftarrow \langle\langle \pi_1(\pi_1 y_2), z\rangle, \pi_2 y_2\rangle$$

$$\text{in } \langle \pi_1(\pi_1 y_1), \langle \pi_2(\pi_1 y_1), \pi_2 y_1\rangle\rangle) : \tau_1 \times (\tau_2' \times \tau_3)$$

$$\stackrel{1\beta}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2(\pi_1 y_2)]$$

$$\text{in } \langle \pi_1(\pi_1 \langle\langle \pi_1(\pi_1 y_2), z\rangle, \pi_2 y_2\rangle),$$

$$\langle \pi_2(\pi_1 \langle\langle \pi_1(\pi_1 y_2), z\rangle, \pi_2 y_2\rangle), \pi_2 \langle\langle \pi_1(\pi_1 y_2), z\rangle, \pi_2 y_2\rangle\rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)$$

$$
\begin{aligned}
&\overset{\mathrm{p}\beta}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2(\pi_1 y_2)] \\
&\qquad\qquad \text{in } \langle \pi_1(\pi_1 y_2), \langle z, \pi_2 y_2 \rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3) \\
&= y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 y_2), \langle M[x \mapsto \pi_2(\pi_1 y_2)], \pi_2 y_2 \rangle\rangle : \tau_1 \times (\tau_2' \times \tau_3)
\end{aligned}
$$

$$
\begin{array}{ccc}
(\tau_1 \times \tau_2) \times \tau_3 & \xrightarrow{a_{\tau_1,\tau_2,\tau_3}} & \tau_1 \times (\tau_2 \times \tau_3) \\
{\scriptstyle (\tau_1 \times \tau_2) \rtimes (x:\tau_3 \vdash M:\tau_3')}\big\downarrow & & \big\downarrow{\scriptstyle \tau_1 \ltimes (\tau_2 \ltimes (x:\tau_3 \vdash M:\tau_3'))} \\
(\tau_1 \times \tau_2) \times \tau_3' & \xrightarrow[a_{\tau_1,\tau_2,\tau_3'}]{} & \tau_1 \times (\tau_2 \times \tau_3')
\end{array}
$$

$$
\begin{aligned}
&(\tau_1 \rtimes (\tau_2 \rtimes (x : \tau_3 \vdash M : \tau_3'))) \circ a_{\tau_1,\tau_2,\tau_3} \\
&\quad = (y_1 : \tau_1 \times (\tau_2 \times \tau_3) \vdash \langle \pi_1 y_1, \langle \pi_1(\pi_2 y), M[x \mapsto (\pi_2(\pi_2 y_1))]\rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3')) \\
&\qquad\quad \circ (y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2 \rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3)) \\
&\quad = y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } y_1 \Leftarrow \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2 \rangle\rangle \\
&\qquad\qquad\quad \text{in } \langle \pi_1 y_1, \langle \pi_1(\pi_2 y), M[x \mapsto (\pi_2(\pi_2 y_1))]\rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3') \\
&\quad \overset{\mathrm{l}\beta}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1 \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2 \rangle\rangle, \\
&\qquad\qquad\quad \langle \pi_1(\pi_2 y), M[x \mapsto (\pi_2(\pi_2 \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 \rangle\rangle))]\rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3') \\
&\quad \overset{\mathrm{p}\beta}{=} y_2 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 y_2), \langle \pi_1(\pi_2 y), M[x \mapsto (\pi_2 y_2))]\rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3')
\end{aligned}
$$

$$
\begin{aligned}
&a_{\tau_1,\tau_2,\tau_3'} \circ ((\tau_1 \times \tau_2) \rtimes (x : \tau_3 \vdash M : \tau_3')) \\
&\quad = (y_2 : (\tau_1 \times \tau_2) \times \tau_3' \vdash \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2 \rangle\rangle : \tau_1 \times (\tau_2 \times \tau_3')) \\
&\qquad\quad \circ (y_1 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2 y_1] \text{ in } \langle \pi_1 y_1, z \rangle : (\tau_1 \times \tau_2) \times \tau_3') \\
&\quad = y_1 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } y_2 \Leftarrow (\text{let } z \Leftarrow M[x \mapsto \pi_2 y_1] \text{ in } \langle \pi_1 y_1, z \rangle) \\
&\qquad\qquad\quad \text{in } \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2 \rangle\rangle : (\tau_1 \times \tau_2) \times \tau_3' \\
&\quad \overset{\mathrm{a}}{=} y_1 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2 y_1]
\end{aligned}
$$

$$\text{in } (\text{let } y_2 \Leftarrow \langle \pi_1 y_1, z\rangle \text{ in } \langle \pi_1(\pi_1 y_2), \langle \pi_2(\pi_1 y_2), \pi_2 y_2\rangle\rangle) : (\tau_1 \times \tau_2) \times \tau_3'$$

$$\stackrel{1\beta}{=} y_1 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2 y_1]$$

$$\text{in } \langle \pi_1(\pi_1\langle \pi_1 y_1, z\rangle), \langle \pi_2(\pi_1\langle \pi_1 y_1, z\rangle), \pi_2\langle \pi_1 y_1, z\rangle\rangle\rangle : (\tau_1 \times \tau_2) \times \tau_3')$$

$$\stackrel{\mathrm{p}\beta}{=} y_1 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2 y_1]$$

$$\text{in } \langle \pi_1(\pi_1 y_1), \langle \pi_2(\pi_1 y_1), z\rangle\rangle : (\tau_1 \times \tau_2) \times \tau_3')$$

$$= y_1 : (\tau_1 \times \tau_2) \times \tau_3 \vdash \langle \pi_1(\pi_1 y_1), \langle \pi_2(\pi_1 y_1), M[x \mapsto \pi_2 y_1]\rangle\rangle : (\tau_1 \times \tau_2) \times \tau_3')$$

Hence a is indeed a natural transformation. □

Lemma 19. *a_{τ_1,τ_2,τ_3}, λ_τ, ρ_τ are isomorphisms.*

Proof. By Lemma 15, $a^{\mathbb{V}}_{\tau_1,\tau_2,\tau_3}$, $\lambda^{\mathbb{V}}_\tau$ and $\rho^{\mathbb{V}}_\tau$ are isomorphisms, and a_{τ_1,τ_2,τ_3}, λ_τ and ρ_τ are their images respectively under the functor J, so they are isomorphisms in $\mathbb{C}$.

□

Hence we can formulate the following proposition.

Proposition 4. *$\mathbb{C}$ is a premonoidal category.*

4.5.6 $\mathbb{C}$ is a symmetric premonoidal category

Define $s_{\tau_1,\tau_2} := (x : \tau_1 \times \tau_2 \vdash \langle \pi_2 x, \pi_1 x\rangle : \tau_2 \times \tau_1)$.

Lemma 20. *s is a natural transformation with components $s_{\tau_1,\tau_2} : \tau_1 \otimes \tau_2 \to \tau_2 \otimes \tau_1$.*

Proof. We are required to prove that the following naturality square commutes.

$$\begin{array}{ccc}
\tau_1 \times \tau_2 & \xrightarrow{(y:\tau_1 \vdash M:\tau_1') \ltimes \tau_2} & \tau_1' \times \tau_2 \\
{\scriptstyle s_{\tau_1,\tau_2}} \downarrow & & \downarrow {\scriptstyle s_{\tau_1',\tau_2}} \\
\tau_2 \times \tau_1 & \xrightarrow[\tau_2 \rtimes (y:\tau_1 \vdash M:\tau_1')]{} & \tau_2 \times \tau_1'
\end{array}$$

Indeed,

$$
\begin{aligned}
& s_{\tau_1',\tau_2} \circ ((y : \tau_1 \vdash M : \tau_1') \ltimes \tau_2) \\
& \quad = (x : \tau_1' \times \tau_2 \vdash \langle \pi_2 x, \pi_1 x\rangle : \tau_2 \times \tau_1') \circ (z : \tau_1 \times \tau_2 \vdash \langle M[y \mapsto \pi_1 z], \pi_2 z\rangle) \\
& \quad = z : \tau_1 \times \tau_2 \vdash \text{let } x \Leftarrow \langle M[y \mapsto \pi_1 z], \pi_2 z\rangle \text{ in } \langle \pi_2 x, \pi_1 x\rangle : \tau_2 \times \tau_1') \\
& \quad \overset{\text{cp}}{=} z : \tau_1 \times \tau_2 \vdash \text{let } x \Leftarrow (\text{let } w \Leftarrow M[y \mapsto \pi_1 z] \text{ in } \langle w, \pi_2 z\rangle) \text{ in } \langle \pi_2 x, \pi_1 x\rangle : \tau_2 \times \tau_1' \\
& \quad \overset{\text{a}}{=} z : \tau_1 \times \tau_2 \vdash \text{let } w \Leftarrow M[y \mapsto \pi_1 z] \text{ in } (\text{let } x \Leftarrow \langle w, \pi_2 z\rangle \text{ in } \langle \pi_2 x, \pi_1 x\rangle) : \tau_2 \times \tau_1' \\
& \quad \overset{\text{l}\beta}{=} z : \tau_1 \times \tau_2 \vdash \text{let } w \Leftarrow M[y \mapsto \pi_1 z] \text{ in } (\langle \pi_2 \langle w, \pi_2 z\rangle, \pi_1 \langle w, \pi_2 z\rangle\rangle) : \tau_2 \times \tau_1' \\
& \quad \overset{\text{p}\beta}{=} z : \tau_1 \times \tau_2 \vdash \text{let } w \Leftarrow M[y \mapsto \pi_1 z] \text{ in } \langle \pi_2 z, w\rangle : \tau_2 \times \tau_1' \\
& \quad \overset{\text{cp}}{=} z : \tau_1 \times \tau_2 \vdash \langle \pi_2 z, M[y \mapsto \pi_1 z]\rangle : \tau_2 \times \tau_1'
\end{aligned}
$$

and

$$
\begin{aligned}
& (\tau_2 \rtimes (y : \tau_1 \vdash M : \tau_1')) \circ s_{\tau_1,\tau_2} \\
& \quad = (z : \tau_2 \times \tau_1 \vdash \langle \pi_1 z, M[y \mapsto \pi_2 z]\rangle : \tau_2 \times \tau_1') \circ (x : \tau_1 \times \tau_2 \vdash \langle \pi_2 x, \pi_1 x\rangle : \tau_2 \times \tau_1) \\
& \quad = x : \tau_1 \times \tau_2 \vdash \text{let } z \Leftarrow \langle \pi_2 x, \pi_1 x\rangle \text{ in } \langle \pi_1 z, M[y \mapsto \pi_2 z]\rangle \\
& \quad \overset{\text{l}\beta}{=} x : \tau_1 \times \tau_2 \vdash \langle \pi_1 \langle \pi_2 x, \pi_1 x\rangle, M[y \mapsto \pi_2 \langle \pi_2 x, \pi_1 x\rangle]\rangle \\
& \quad \overset{\text{p}\beta}{=} x : \tau_1 \times \tau_2 \vdash \langle \pi_2 x, M[y \mapsto \pi_1 x]\rangle : \tau_2 \times \tau_1'.
\end{aligned}
$$

Hence the above square commutes. By logical symmetry, s_{τ_1,τ_2} is also natural in the second position, so it is indeed natural as required to prove. □

Lemma 21. *For any* τ_1, τ_2*,* s_{τ_1,τ_2} *is central.*

Proof. It is a value, so it is central by Lemma 14. □

Lemma 22. *If we regard* $\mathbb{V}$ *as a symmetric premonoidal category with* $\times$ *the product,* *as in Corollary 1, then the symmetry is* $s^{\mathbb{V}}_{\tau_1,\tau_2} = (x : \tau_1 \times \tau_2 \vdash \langle \pi_2 x, \pi_1 x\rangle : \tau_2 \times \tau_1)$*, so we have* $s_{\tau_1,\tau_2} = J s^{\mathbb{V}}_{\tau_1,\tau_2}$. □

Lemma 23. s_{τ_1,τ_2} *is an isomorphism.*

Proof. By Lemma 22 s_{τ_1,τ_2} is the image of an isomorphism under the functor J, so it is also an isomorphism. □

Lemma 24. *The following diagram commutes:*

$$\begin{array}{ccccccc}
(\tau_1 \times \tau_2) \times \tau_3 & \xrightarrow{a_{\tau_1,\tau_2,\tau_3}} & \tau_1 \times (\tau_2 \times \tau_3) & \xrightarrow{s_{\tau_1,\tau_2 \times \tau_3}} & (\tau_2 \times \tau_3) \times \tau_1 & \xrightarrow{s_{\tau_2,\tau_3} \ltimes \tau_1} & (\tau_3 \times \tau_2) \times \tau_1 \\
 & & & & & & \downarrow a_{\tau_3,\tau_2,\tau_1} \\
 & & & & & & \tau_3 \times (\tau_2 \times \tau_1) \\
 & & & & & & \downarrow \tau_3 \rtimes s_{\tau_2,\tau_1} \\
 & & & \searrow s_{\tau_1 \times \tau_2,\tau_3} & & & \tau_3 \times (\tau_1 \times \tau_2)
\end{array}$$

Proof. We have seen above that J maps the symmetric premonoidal structure of $\mathbb{V}$ to the claimed symmetric premonoidal structure of $\mathbb{C}$, and J is a functor, so it preserves commuting diagrams.

The above diagram is the symmetry condition, so the corresponding diagram holds in $\mathbb{V}$, so this diagram holds in $\mathbb{C}$. □

Hence we can deduce the following two propositions.

Proposition 5. $\mathbb{C}$ *is a symmetric premonoidal category.*

Proposition 6. J *is an identity-on-object functor that strictly preserves symmetric premonoidal structure.*

4.5.7 $\mathbb{V} \xrightarrow{J} \mathbb{C}$ is a closed Freyd-category

Define

$$(\tau_1 \Rightarrow \tau_2) := (\tau_1 \to \tau_2)$$

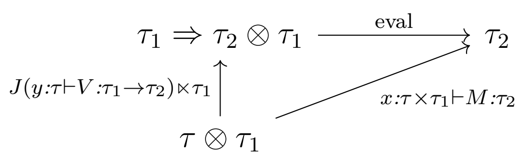
$$\text{eval} : (\tau_1 \Rightarrow \tau_2) \otimes \tau_1 \to \tau_2$$

$$\text{eval} := (x : (\tau_1 \to \tau_2) \times \tau_1 \vdash (\pi_1 x)(\pi_2 x) : \tau_2)$$

And for any $(x : \tau \times \tau_1 \vdash M : \tau_2)$ in $\mathbb{C}$, let

$$\Lambda(x : \tau \times \tau_1 \vdash M : \tau_2) := y : \tau \vdash \lambda z : \tau_1.M[x \mapsto \langle y, z\rangle] : \tau_1 \to \tau_2.$$

Lemma 25. *This is the unique* $y : \tau \vdash V : \tau_1 \to \tau_2$ *such that*



commutes.

Proof. For any $y : \tau \vdash V : \tau_1 \to \tau_2$,

$$\begin{aligned}
&\text{eval} \circ (J(y : \tau \vdash V : \tau_1 \to \tau_2) \ltimes \tau_1)\\
&\quad = (w : (\tau_1 \to \tau_2) \times \tau_1 \vdash (\pi_1 w)(\pi_2 w) : \tau_2) \circ (z : \tau \times \tau_1 \vdash \langle V[y \mapsto \pi_1 z], \pi_2 z\rangle)\\
&\quad = q : \tau \times \tau_1 \vdash \text{let } w \Leftarrow \langle V[y \mapsto \pi_1 q], \pi_2 q\rangle \text{ in } (\pi_1 w)(\pi_2 w) : \tau_2\\
&\quad \stackrel{\text{l}\beta}{=} q : \tau \times \tau_1 \vdash (\pi_1\langle V[y \mapsto \pi_1 q], \pi_2 q\rangle)(\pi_2\langle V[y \mapsto \pi_1 q], \pi_2 q\rangle) : \tau_2)\\
&\quad \stackrel{\text{p}\beta}{=} q : \tau \times \tau_1 \vdash (V[y \mapsto \pi_1 q])(\pi_2 q) : \tau_2.
\end{aligned}$$

So for $V = \lambda z : \tau_1.M[x \mapsto \langle y, z\rangle]$,

$$\begin{aligned}
&\text{eval} \circ (J(y : \tau \vdash V : \tau_1 \to \tau_2) \ltimes \tau_1)\\
&\quad = q : \tau \times \tau_1 \vdash (\lambda z : \tau_1.M[x \mapsto \langle y, z\rangle][y \mapsto \pi_1 q])(\pi_2 q) : \tau_2\\
&\quad = q : \tau \times \tau_1 \vdash (\lambda z : \tau_1.M[x \mapsto \langle \pi_1 q, z\rangle])(\pi_2 q) : \tau_2\\
&\quad \stackrel{\text{f}\beta}{=} q : \tau \times \tau_1 \vdash (M[x \mapsto \langle \pi_1 q, \pi_2 q\rangle]) : \tau_2
\end{aligned}$$

$$\overset{\text{p}\eta}{=} q : \tau \times \tau_1 \vdash (M[x \mapsto q]) : \tau_2 = \tau \times \tau_1 \vdash M : \tau_2$$

as required.

However, it is unique with this property (with respect to the congruence from above), because if $\text{eval} \circ (J(y : \tau \vdash V : \tau_1 \to \tau_2) \ltimes \tau_1) = x : \tau \times \tau_1 \vdash M : \tau_2$, then:

$$\begin{aligned}
&y_0 : \tau \vdash \lambda z.M[x \mapsto \langle y_0, z\rangle] \\
&\quad = y_0 : \tau \vdash \lambda z.(\text{eval} \circ (J(y : \tau \vdash V : \tau_1 \to \tau_2) \ltimes \tau_1))[x \mapsto \langle y_0, z\rangle] \\
&\quad = y_0 : \tau \vdash \lambda z.((V[y \mapsto \pi_1 x])(\pi_2 x))[x \mapsto \langle y_0, z\rangle] \\
&\quad = y_0 : \tau \vdash \lambda z.(V[y \mapsto \pi_1\langle y_0, z\rangle])(\pi_2\langle y_0, z\rangle) \\
&\quad \overset{\text{p}\beta}{=} y_0 : \tau \vdash \lambda z.(V[y \mapsto y_0])z) \\
&\quad = y : \tau \vdash \lambda z.Vz \\
&\quad \overset{\text{f}\eta}{=} y : \tau \vdash V : \tau_1 \to \tau_2.
\end{aligned}$$

Hence $\Lambda(x : \tau \times \tau_1 \vdash M : \tau_2)$ is indeed the unique value with that property. □

Theorem 12. $\mathbb{V} \xrightarrow{J} \mathbb{C}$ *is a closed Freyd-category.*

Proof. By Proposition 3 $\mathbb{V}$ is a category with finite products, by Proposition 5 $\mathbb{C}$ is a symmetric premonoidal category, and by Proposition 6 J is an identity-on-objects functor strictly preserving the symmetric premonoidal structure and it maps central morphisms to central morphisms, so $\mathbb{V} \xrightarrow{J} \mathbb{C}$ is a Freyd-category. Furthermore, by Lemma 25, it is a closed Freyd-category. □

4.6 Free property

Definition 33 (Strict closed Freyd-functor)**.** *For a closed Freyd-categories* $\mathbb{V}_1 \xrightarrow{J_1} \mathbb{C}_1$, $\mathbb{V}_2 \xrightarrow{J_2} \mathbb{C}_2$ *let us call* $F = (F_{\mathbb{V}}, F_{\mathbb{C}})$ *a* strict closed Freyd-functor *if:*

(C1) $F_{\mathbb{V}} : \mathbb{V}_1 \to \mathbb{V}_2$ *is a functor that strictly preserves finite products;*

(C2) $F_{\mathbb{C}} : \mathbb{C}_1 \to \mathbb{C}_2$ *is a functor that strictly preserves symmetric premonoidal structure;*

(C3) $F_{\mathbb{V}}$ *and* $F_{\mathbb{C}}$ *strictly preserve the closed structure;*

(C4) The following diagram commutes:

$$\begin{CD} \mathbb{V}_1 @>{J_1}>> \mathbb{C}_1 \\ @V{F_{\mathbb{V}}}VV @VV{F_{\mathbb{C}}}V \\ \mathbb{V}_2 @>>{J_2}> \mathbb{C}_2 \end{CD}$$

▲

Definition 34 (Free closed Freyd-category over a signature)**.** *Given a signature* $\mathcal{S} = (\mathcal{S}_{type}, \mathcal{S}_{const})$ *a closed Freyd-category* $\mathcal{F}[\mathcal{S}] = \mathbb{V}_{\mathcal{S}} \xrightarrow{J_{\mathcal{S}}} \mathbb{C}_{\mathcal{S}}$ *is* free *over* $\mathcal{S}$ *iff there exists an interpretation* ι *of* $\mathcal{S}$ *in* $\mathcal{F}[\mathcal{S}]$ *such that for any Freyd-category* $\mathbb{V} \xrightarrow{J} \mathbb{C}$*, and any interpretation* F *of* $\mathcal{S}$ *in* $\mathbb{V} \xrightarrow{J} \mathbb{C}$*, there is a unique strict closed Freyd-functor* $F^{\#}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} (\mathbb{V}_{\mathcal{S}} \xrightarrow{J_{\mathcal{S}}} \mathbb{C}_{\mathcal{S}}) & \xrightarrow{F^{\#}} & (\mathbb{V} \xrightarrow{J} \mathbb{C}) \\ {\scriptstyle\iota}\uparrow & \nearrow_{F} & \\ B & & \end{array} \tag{C5}$$

i.e, for any $\beta \in \mathcal{S}_{type}$*,* $F^{\#}_{\mathbb{V}}(\iota(\beta)) = F(\beta)$*, any* $(c_{prim}, \tau) \in \mathcal{S}_{prim}$*,* $F^{\#}_{\mathbb{V}}(\iota(c_{prim})) = F_{\mathbb{V}}(c_{prim})$*, and any* $(c_{efop}, \tau) \in \mathcal{S}_{efop}$*,* $F^{\#}_{\mathbb{C}}(\iota(c_{efop})) = F_{\mathbb{C}}(c_{efop})$*.* ▲

This section contains the proof of the following theorem, the key theorem of this dissertation.

Theorem 13. *Given a signature* $\mathcal{S}$*, the syntactic closed Freyd-category of the* *computational lambda calculus with that signature,* $\mathbb{V}_{\mathcal{S}} \xrightarrow{J_{\mathcal{S}}} \mathbb{C}_{\mathcal{S}}$*, is the free closed Freyd-category over that signature.*

In particular, given a closed Freyd-category $\mathbb{V} \xrightarrow{J} \mathbb{C}$ and an interpretation F of $\mathcal{S}$ in $\mathbb{V} \xrightarrow{J} \mathbb{C}$ we can define $F^{\#} = (F^{\#}_{\mathbb{V}}, F^{\#}_{\mathbb{C}})$ as follows. On objects:

$$F^{\#}_{\mathbb{V}}(1) = F^{\#}_{\mathbb{C}}(1) = 1 \tag{O1}$$

$$F^{\#}_{\mathbb{V}}(\beta) = F^{\#}_{\mathbb{C}}(\beta) = F(\beta) \qquad \text{for } \beta \text{ in } \mathcal{S}_{type} \tag{O2}$$

$$F^{\#}_{\mathbb{V}}(\tau_1 \times \tau_2) = F^{\#}_{\mathbb{C}}(\tau_1 \times \tau_2) = F^{\#}_{\mathbb{V}}(\tau_1) \times F^{\#}_{\mathbb{V}}(\tau_2) = F^{\#}_{\mathbb{C}}(\tau_1) \otimes F^{\#}_{\mathbb{C}}(\tau_2) \tag{O3}$$

$$F^{\#}_{\mathbb{V}}(\tau_1 \to \tau_2) = F^{\#}_{\mathbb{C}}(\tau_1 \to \tau_2) = F^{\#}_{\mathbb{V}}(\tau_1) \Rightarrow F^{\#}_{\mathbb{V}}(\tau_2) = F^{\#}_{\mathbb{C}}(\tau_1) \Rightarrow F^{\#}_{\mathbb{C}}(\tau_2) \tag{O4}$$

On morphisms of $\mathbb{V}_{\mathcal{S}}$:

$$F^{\#}_{\mathbb{V}}(x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i) = \pi_i \tag{MV1}$$

$$F^{\#}_{\mathbb{V}}(\Gamma \vdash () : 1) = ! \tag{MV2}$$

$$F^{\#}_{\mathbb{V}}(\Gamma \vdash \pi_i V : \tau_i) = \pi_i \circ F^{\#}_{\mathbb{V}}(\Gamma \vdash V : \tau_1 \times \tau_2) \tag{MV3}$$

$$F^{\#}_{\mathbb{V}}(\Gamma \vdash \langle V_1, V_2 \rangle : \tau_1 \times \tau_2) = \langle F^{\#}_{\mathbb{V}}(\Gamma \vdash V_1 : \tau_1), F^{\#}_{\mathbb{V}}(\Gamma \vdash V_2 : \tau_2) \rangle \tag{MV4}$$

$$\begin{aligned} F^{\#}_{\mathbb{V}}(\Gamma \vdash \text{let } x \Leftarrow V_1 \text{ in } V_2 : \tau_2) = {} & F^{\#}_{\mathbb{V}}(\Gamma, x : \tau_1 \vdash V_2 : \tau_2) \\ & \circ (\text{id} \times F^{\#}_{\mathbb{V}}(\Gamma \vdash V_1 : \tau_1)) \circ \Delta \end{aligned} \tag{MV5}$$

$$F^{\#}_{\mathbb{V}}(\Gamma \vdash \lambda x.M : \tau_1 \to \tau_2) = \Lambda \left(F^{\#}_{\mathbb{C}}(\Gamma, x : \tau_1 \vdash M : \tau_2) \right) \tag{MV6}$$

$$F^{\#}_{\mathbb{V}}(\Gamma \vdash c_{prim} : \tau) = F_{\mathbb{V}}(c_{prim}) \circ ! \tag{MV7}$$

On morphisms of $\mathbb{C}_{\mathcal{S}}$:

$$F^{\#}_{\mathbb{C}}(\Gamma \vdash V : \tau) = J F^{\#}_{\mathbb{V}}(\Gamma \vdash V : \tau) \tag{MC1}$$

$$\begin{aligned} F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2 : \tau_2) = {} & F^{\#}_{\mathbb{C}}(\Gamma, x : \tau_1 \vdash M_2 : \tau_2) \\ & \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1)) \circ J\Delta \end{aligned} \tag{MC2}$$

$$F^{\#}_{\mathbb{C}}(\Gamma \vdash \pi_i M : \tau_i) = J\pi_i \circ F^{\#}_{\mathbb{C}}(\Gamma \vdash M) \tag{MC3}$$

$$F^{\#}_{\mathbb{C}}(\Gamma \vdash \langle M_1, M_2\rangle : \tau_1 \times \tau_2) = (\mathrm{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2)) \circ (F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1) \otimes \mathrm{id}) \circ J\Delta \tag{MC4}$$

$$\begin{aligned} F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 M_2 : \tau_2) = \mathrm{eval} \circ (\mathrm{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2 : \tau_1)) \\ \circ (F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1 \to \tau_2) \otimes \mathrm{id}) \circ J\Delta \end{aligned} \tag{MC5}$$

$$F^{\#}_{\mathbb{C}}(\Gamma \vdash c_{efop} : \tau) = F_{\mathbb{C}}(c_{efop}) \circ ! \tag{MC6}$$

Note that syntactically this is the same definition as the interpretation of the computational lambda calculus in a Freyd-category from Figure 4.3, but conceptually this is a mapping from objects and morphism of the syntactic category, not types and terms of the computational lambda calculus.

4.6.1 $F^{\#}$ is a strict closed Freyd-functor

Lemma 26. *$F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ are functors.*

Proof. We defined $F^{\#}_{\mathbb{V}}(A)$, $F^{\#}_{\mathbb{C}}(A)$ for all objects of $\mathbb{V}_{\mathcal{S}}$ and $\mathbb{C}_{\mathcal{S}}$ and $F^{\#}_{\mathbb{V}}(f_{\mathbb{V}})$, $F^{\#}_{\mathbb{C}}(f_{\mathbb{C}})$ for all morphisms.

Furthermore, we have seen that the interpretation of λ_C in a closed Freyd-category is sound with respect to the equations we quotient with, so $F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ are **well-defined** with respect to the quotienting.

Furthermore, they respect identities:

$$F^{\#}_{\mathbb{V}}(x : \tau \vdash x : \tau) = \mathrm{id}^{\mathbb{V}}_{\tau}$$

$$F^{\#}_{\mathbb{C}}(x : \tau \vdash x : \tau) = JF^{\#}_{\mathbb{V}}(x : \tau \vdash x : \tau) = J\mathrm{id}^{\mathbb{V}}_{\tau} = \mathrm{id}^{\mathbb{C}}_{\tau}$$

and composition:

$$
\begin{aligned}
&F^{\#}_{\mathbb{V}}((y:\tau_2 \vdash V_2:\tau_3)\circ(x:\tau_1 \vdash V_1:\tau_2))\\
&\quad = F^{\#}_{\mathbb{V}}(x:\tau_1 \vdash \text{let } y \Leftarrow V_1 \text{ in } V_2:\tau_3)\\
&\quad = F^{\#}_{\mathbb{V}}(x:\tau_1, y:\tau_2 \vdash V_2:\tau_3)\circ(\text{id}\times F^{\#}_{\mathbb{V}}(x:\tau_1 \vdash V_1:\tau_2))\circ\Delta\\
&\quad \overset{\text{w}}{=} F^{\#}_{\mathbb{V}}(y:\tau_2 \vdash V_2:\tau_3)\circ\pi_2\circ(\text{id}\times F^{\#}_{\mathbb{V}}(x:\tau_1 \vdash V_1:\tau_2))\circ\Delta\\
&\quad = F^{\#}_{\mathbb{V}}(y:\tau_2 \vdash V_2:\tau_3)\circ F^{\#}_{\mathbb{V}}(x:\tau_1 \vdash V_1:\tau_2)
\end{aligned}
$$

$$
\begin{aligned}
&F^{\#}_{\mathbb{C}}((y:\tau_2 \vdash M_2:\tau_3)\circ(x:\tau_1 \vdash M_1:\tau_2))\\
&\quad = F^{\#}_{\mathbb{C}}(x:\tau_1 \vdash \text{let } y \Leftarrow M_1 \text{ in } M_2:\tau_3)\\
&\quad = F^{\#}_{\mathbb{C}}(x:\tau_1, y:\tau_2 \vdash M_2:\tau_3)\circ(\text{id}\otimes F^{\#}_{\mathbb{C}}(x:\tau_1 \vdash M_1:\tau_2))\circ J\Delta\\
&\quad \overset{\text{w}}{=} F^{\#}_{\mathbb{C}}(y:\tau_2 \vdash M_2:\tau_3)\circ J\pi_2\circ(\text{id}\otimes F^{\#}_{\mathbb{C}}(x:\tau_1 \vdash M_1:\tau_2))\circ J\Delta\\
&\quad = F^{\#}_{\mathbb{C}}(y:\tau_2 \vdash M_2:\tau_3)\circ F^{\#}_{\mathbb{C}}(x:\tau_1 \vdash M_1:\tau_2)\circ J\pi_2\circ J\Delta\\
&\quad = F^{\#}_{\mathbb{C}}(y:\tau_2 \vdash M_2:\tau_3)\circ F^{\#}_{\mathbb{C}}(x:\tau_1 \vdash M_1:\tau_2)
\end{aligned}
$$

Hence $F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ are indeed functors, as required. □

Lemma 27. *$F^{\#}_{\mathbb{V}}$ strictly preserves finite products.*

Proof. It strictly preserves the terminal object by (O1). It strictly preserves the unique morphism into the terminal object by (MV2). It strictly preserves the product object by (O3). It strictly preserves projections by (MV3) and pairing by (MV4). □

Lemma 28. *$F^{\#}_{\mathbb{C}}$ strictly preserves symmetric premonoidal structure.*

Proof. It strictly preserves $- \otimes =$ on objects by (O3).

$$F^{\#}_{\mathbb{C}}(\tau \rtimes (x:\tau_1 \vdash M:\tau_2))$$

$$
\begin{aligned}
&= F_{\mathbb{C}}^{\#}(y : \tau \times \tau_1 \vdash \langle \pi_1 y, M[x \mapsto \pi_2 y] \rangle : \tau_\times \tau_2) \\
&= F_{\mathbb{C}}^{\#}(y : \tau \times \tau_1 \vdash \text{let } z \Leftarrow M[x \mapsto \pi_2 y] \text{ in } \langle \pi_1 y, z \rangle : \tau_\times \tau_2) \\
&= F_{\mathbb{C}}^{\#}(y : \tau \times \tau_1, z : \tau_2 \vdash \langle \pi_1 y, z \rangle : \tau_\times \tau_2) \circ (\mathrm{id} \otimes F_{\mathbb{C}}^{\#}(y : \tau \times \tau_1 \vdash M[x \mapsto \pi_2 y])) \\
&\quad \circ J\Delta \\
&= J\langle \pi_1 \circ \pi_1, \pi_2 \rangle \circ \Big(\mathrm{id} \otimes (F_{\mathbb{C}}^{\#}(x : \tau_1 \vdash M : \tau_2) \circ J\pi_2)\Big) \circ J\Delta \\
&= (J\pi_1 \otimes \mathrm{id}) \circ \Big(\mathrm{id} \otimes F_{\mathbb{C}}^{\#}(x : \tau_1 \vdash M : \tau_2)\Big) \circ (\mathrm{id} \otimes J\pi_2) \circ J\Delta \\
&= (\mathrm{id} \otimes F_{\mathbb{C}}^{\#}(x : \tau_1 \vdash M : \tau_2)) \circ (J\pi_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes J\pi_2) \circ J\Delta \\
&= (\mathrm{id} \otimes F_{\mathbb{C}}^{\#}(x : \tau_1 \vdash M : \tau_2))
\end{aligned}
$$

Similarly, it also strictly preserves $- \ltimes \tau$.

$F_{\mathbb{C}}^{\#}(I) = 1$ by (O1).

$$F_{\mathbb{C}}^{\#}(a_{X,Y,Z}) \stackrel{*}{=} JF_{\mathbb{V}}^{\#}(a_{X,Y,Z}) \stackrel{\dagger}{=} Ja^{\mathbb{V}}_{X,Y,Z} = a^{\mathbb{C}}_{X,Y,Z}$$

where $*$ hold because $F_{\mathbb{V}}^{\#}$ strictly preserves finite products by Lemma 27 and † holds because J strictly preserves premonoidal structure.

Similarly

$$
\begin{aligned}
F_{\mathbb{C}}^{\#}(\lambda_X) &= \lambda_X^{\mathbb{C}} \\
F_{\mathbb{C}}^{\#}(\rho_X) &= \rho_X^{\mathbb{C}} \\
F_{\mathbb{C}}^{\#}(s_{X,Y}) &= s_{X,Y}^{\mathbb{C}}.
\end{aligned}
$$

So indeed $F_{\mathbb{C}}^{\#}$ strictly preserves the symmetric premonoidal structure. □

Lemma 29. *$F_{\mathbb{V}}^{\#}$ and $F_{\mathbb{C}}^{\#}$ strictly preserve the closed structure.*

Proof. The exponential object is preserved by (O4).

$$\begin{aligned}
&F^{\#}_{\mathbb{C}}(\text{eval}) \\
&\quad = F^{\#}_{\mathbb{C}}(x : (\tau_1 \to \tau_2) \times \tau_1 \vdash (\pi_1 x)(\pi_2 x) : \tau_2) \\
&\quad = \text{eval} \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(x : (\tau_1 \to \tau_2) \times \tau_1 \vdash \pi_2 x : \tau_1)) \\
&\qquad\quad \circ (F^{\#}_{\mathbb{C}}(x : (\tau_1 \to \tau_2) \times \tau_1 \vdash \pi_1 x : \tau_1 \to \tau_2) \otimes \text{id}) \circ J\Delta \\
&\quad = \text{eval} \circ (\text{id} \otimes J\pi_1) \circ (J\pi_2 \otimes \text{id}) \circ J\Delta \\
&\quad = \text{eval}
\end{aligned}$$

$$\begin{aligned}
&F^{\#}_{\mathbb{V}}(\Lambda(x : \tau \times \tau_1 \vdash M : \tau_2)) \\
&\quad = F^{\#}_{\mathbb{V}}(y : \tau \vdash \lambda z.M[x \mapsto \langle y, z \rangle] : \tau_1 \to \tau_2) \\
&\quad = \Lambda(F^{\#}_{\mathbb{C}}(y : \tau, z : \tau_1 \vdash M[x \mapsto \langle y, z \rangle] : \tau_2)) \\
&\quad = \Lambda(F^{\#}_{\mathbb{C}}(x : \tau \times \tau_1 \vdash M : \tau_2))
\end{aligned}$$

So $F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ strictly preserve the closed structure, as required. □

Lemma 30. *$F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ satisfy (C4).*

Proof. Using (MC1),

$$F^{\#}_{\mathbb{C}}(J_{\mathcal{S}}(x : \tau_1 \vdash V : \tau_2)) = F^{\#}_{\mathbb{C}}(x : \tau_1 \vdash V : \tau_2) = JF^{\#}_{\mathbb{V}}(x : \tau_1 \vdash V : \tau_2).$$

□

Lemma 31. *$F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ satisfy (C5).*

Proof. For any $\beta \in \mathcal{S}_{type}$,

$$F^{\#}_{\mathbb{V}}(\iota\beta) = F^{\#}_{\mathbb{V}}(\beta) = F(\beta)$$

by (O2).

Similarly, for any $(\tau, c_{prim}) \in \mathcal{S}_{prim}$, using (MV7),

$$F^{\#}_{\mathbb{V}}(\iota(c_{prim})) = F^{\#}_{\mathbb{V}}(\vdash c_{prim} : \tau) = F(c_{prim}),$$

and any $(\tau, c_{efop}) \in \mathcal{S}_{efop}$, using (MV6),

$$F^{\#}_{\mathbb{C}}(\iota(c_{efop})) = F^{\#}_{\mathbb{C}}(\vdash c_{efop} : \tau) = F(c_{efop}).$$

□

Hence $F^{\#}$ satisfies all of the requirements $(C1)$, $(C2)$, $(C3)$, $(C4)$ and $(C5)$.

4.6.2 $F^{\#}$ is unique

We are going to prove that $F^{\#}$ as defined above is unique by showing that each of the above rules has to hold for $F^{\#}$ with the required properties.

For rules O1-O4 we will only argue for $F^{\#}_{\mathbb{V}}$, but using that J_1 and J_2 are identities-on-object, and by $C4$, $F^{\#}_{\mathbb{C}}$ has to behave the same way.

O1 has to hold because $F^{\#}_{\mathbb{V}}$ strictly preserves finite products.

O2 has to hold by $(C5)$.

O3 has to hold to because $F^{\#}_{\mathbb{V}}$ strictly preserves finite products.

O4 has to hold to because $F^{\#}_{\mathbb{V}}$ strictly preserves the closed structure.

MV1 has to hold because $F^{\#}_{\mathbb{V}}$ strictly preserves finite products, and it is a functor, so

$$F^{\#}_{\mathbb{V}}(x : ((\tau_1 \times \tau_2) \cdots \times \tau_n) \vdash \pi_i x : \tau_i)$$

$$= \pi_i \circ F^{\#}_{\mathbb{V}}(x : ((\tau_1 \times \tau_2) \cdots \times \tau_n) \vdash x : ((\tau_1 \times \tau_2) \cdots \times \tau_n))$$

$$= \pi_i.$$

MV2 has to hold because $F^{\#}_{\mathbb{V}}$ strictly preserves the unique morphism into the terminal object.

MV3 has to hold because $F^{\#}_{\mathbb{V}}$ strictly preserves finite products, so $F^{\#}_{\mathbb{V}}(x : \tau_1 \times \tau_2 \vdash \pi_i x : \tau_i) = \pi_i$, so using the substitution lemma,

$$\begin{aligned} &F^{\#}_{\mathbb{V}}(\Gamma \vdash (\pi_i x)[x \mapsto V] : \tau_i) \\ &\quad = F^{\#}_{\mathbb{V}}(x : \tau_1 \times \tau_2 \vdash \pi_i x : \tau_i) \circ F^{\#}_{\mathbb{V}}(\Gamma \vdash V) \\ &\quad = \pi_i \circ F^{\#}_{\mathbb{V}}(\Gamma \vdash V). \end{aligned}$$

MV4 has to hold because $F^{\#}_{\mathbb{V}}$ strictly preserves finite products.

MV5 has to hold, because

$$\begin{aligned} &F^{\#}_{\mathbb{V}}(\Gamma \vdash \text{let } x \Leftarrow V_1 \text{ in } V_2) \\ &\quad = F^{\#}_{\mathbb{V}}(\Gamma \vdash V_2[x \mapsto V_1]) \\ &\quad = F^{\#}_{\mathbb{V}}(\Gamma \vdash V_2[\Gamma \mapsto \Gamma, x \mapsto V_1]) \\ &\quad = F^{\#}_{\mathbb{V}}(\Gamma \vdash \text{let } y \Leftarrow \langle \Gamma, V_1 \rangle \text{ in } V_2) \\ &\quad = F^{\#}_{\mathbb{V}}((\Gamma, x : \tau_1 \vdash V_2) \circ (\Gamma \vdash \langle \Gamma, V_1 \rangle : \tau \times \tau_1)) \\ &\quad = F^{\#}_{\mathbb{V}}(\Gamma, x : \tau_1 \vdash V_2) \circ F^{\#}_{\mathbb{V}}(\Gamma \vdash \langle \Gamma, V_1 \rangle : \tau \times \tau_1) \\ &\quad = F^{\#}_{\mathbb{V}}(\Gamma, x : \tau_1 \vdash V_2) \circ \langle \text{id}, F^{\#}_{\mathbb{V}}(\Gamma \vdash V_1 : \tau_1) \rangle. \end{aligned}$$

MV6 has to hold because $F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ strictly preserves closed structure, so

$$F^{\#}_{\mathbb{V}}(\Gamma \vdash \lambda x.M : \tau_1 \to \tau_2) = \Lambda(F^{\#}_{\mathbb{C}}(\Gamma, x : \tau_1 \vdash M : \tau_2)).$$

MV7 has to hold by $(C5)$.

MC1 has to hold by $(C4)$.

MC2 has to hold because

$$\begin{aligned}
&F^{\#}_{\mathbb{C}}(p : \tau \vdash \text{let } x \Leftarrow M_1 \text{ in } M_2[q \mapsto \langle p, x\rangle] \\
&\quad = F^{\#}_{\mathbb{C}}(p : \tau \vdash \text{let } x \Leftarrow M_1 \text{ in let } q \Leftarrow \langle p, x\rangle \text{ in } M_2) \\
&\quad \stackrel{\text{a}}{=} F^{\#}_{\mathbb{C}}(p : \tau \vdash \text{let } q \Leftarrow (\text{let } x \Leftarrow M_1 \text{ in } \langle p, x\rangle) \text{ in } M_2) \\
&\quad \stackrel{\text{cp}}{=} F^{\#}_{\mathbb{C}}(p : \tau \vdash \text{let } q \Leftarrow \langle p, M_1\rangle \text{ in } M_2) \\
&\quad = F^{\#}_{\mathbb{C}}((q : \tau \times \tau_1 \vdash M_2 : \tau_2) \circ (p : \tau \vdash \langle p, M_1\rangle : \tau \times \tau_1)) \\
&\quad = F^{\#}_{\mathbb{C}}(q : \tau \times \tau_1 \vdash M_2 : \tau_2) \circ F^{\#}_{\mathbb{C}}(p : \tau \vdash \langle p, M_1\rangle : \tau \times \tau_1) \\
&\quad = F^{\#}_{\mathbb{C}}(q : \tau \times \tau_1 \vdash M_2 : \tau_2) \\
&\qquad\quad \circ F^{\#}_{\mathbb{C}}(r : \tau \vdash \langle \pi_1\langle r, r\rangle, M_1[p \mapsto \pi_2\langle r, r\rangle]\rangle : \tau \times \tau_1) \\
&\quad = F^{\#}_{\mathbb{C}}(q : \tau \times \tau_1 \vdash M_2 : \tau_2) \\
&\qquad\quad \circ F^{\#}_{\mathbb{C}}(r : \tau \vdash \text{let } t \Leftarrow \langle r, r\rangle \text{ in } \langle \pi_1 t, M[p \mapsto \pi_2 t]\rangle) \\
&\quad = F^{\#}_{\mathbb{C}}(q : \tau \times \tau_1 \vdash M_2 : \tau_2) \circ F^{\#}_{\mathbb{C}}(t : \tau \times \tau \vdash \langle \pi_1 t, M[p \mapsto \pi_2 t]\rangle) \\
&\qquad\quad \circ F^{\#}_{\mathbb{C}}(r : \tau \vdash \langle r, r\rangle) \\
&\quad = F^{\#}_{\mathbb{C}}(q : \tau \times \tau_1 \vdash M_2 : \tau_2) \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\tau \vdash M_1 : \tau_1)) \circ J\Delta.
\end{aligned}$$

MC3 has to hold because

$$F^{\#}_{\mathbb{C}}(\Gamma \vdash \pi_i M)$$

$$\overset{\text{cpr}}{=} F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } y \Leftarrow M \text{ in } \pi_i y)$$

$$\overset{*}{=} F^{\#}_{\mathbb{C}}(\Gamma, y : \tau_1 \times \tau_2 \vdash \pi_i y : \tau_i) \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M : \tau_1 \times \tau_2)) \circ J\Delta$$

$$= J\pi_i \circ J\pi_2 \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M : \tau_1 \times \tau_2)) \circ J\Delta$$

$$= J\pi_i \circ F^{\#}_{\mathbb{C}}(\Gamma \vdash M : \tau_1 \times \tau_2) \circ J\pi_2 \circ J\Delta$$

$$= J\pi_i \circ F^{\#}_{\mathbb{C}}(\Gamma \vdash M : \tau_1 \times \tau_2)$$

where $*$ holds as (MC2) holds.

MC4 has to hold because

$$F^{\#}_{\mathbb{C}}(\Gamma \vdash \langle M_1, M_2 \rangle : \tau_1 \times \tau_2)$$

$$\overset{\text{cp}}{=} F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in let } y \Leftarrow M_2 \text{ in } \langle x, y \rangle : \tau_1 \times \tau_2)$$

$$= F^{\#}_{\mathbb{C}}(\Gamma, x : \tau_1 \vdash \text{let } y \Leftarrow M_2 \text{ in } \langle x, y \rangle : \tau_1 \times \tau_2) \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1))$$
$$\circ J\Delta$$

$$= F^{\#}_{\mathbb{C}}(\Gamma, x : \tau_1, y : \tau_2 \vdash \langle x, y \rangle : \tau_1 \times \tau_2) \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma, x : \tau_1 \vdash M_2)) \circ J\Delta$$
$$\circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1)) \circ J\Delta$$

$$= J\langle \pi_2 \circ \pi_1, \pi_2 \rangle \circ (\text{id} \otimes (F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2) \circ J\pi_1)) \circ J\Delta$$
$$\circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1)) \circ J\Delta$$

$$= (J\pi_2 \otimes \text{id}) \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2)) \circ (\text{id} \otimes J\pi_1) \circ J\Delta$$
$$\circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1)) \circ J\Delta$$

$$= (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2)) \circ J\langle \pi_2, \pi_1 \rangle \circ (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1)) \circ J\Delta$$

$$\overset{*}{=} (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2)) \circ (F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1) \otimes \text{id}) \circ J\langle \pi_2, \pi_1 \rangle \circ J\Delta$$

$$= (\text{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2)) \circ (F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1) \otimes \text{id}) \circ J\Delta$$

where $*$ holds because $J\langle \pi_2, \pi_1 \rangle = s_{\tau_1, \tau_2}$.

MC5 has to hold because the closed structure is strictly preserved, so

$$F^{\#}_{\mathbb{C}}(x : (\tau_1 \to \tau_2) \times \tau_1 \vdash (\pi_1 x)(\pi_2 x) : \tau_2) = \mathrm{eval}$$

so we need

$$\begin{aligned}
&F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 M_2 : \tau_2) \\
&\quad \stackrel{\text{ca}}{=} F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in let } y \Leftarrow M_2 \text{ in } xy : \tau_2) \\
&\quad = F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } x \Leftarrow M_1 \text{ in let } y \Leftarrow M_2 \text{ in let } z \Leftarrow \langle x, y\rangle \text{ in } (\pi_1 z)(\pi_2 z) : \tau_2) \\
&\quad = F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } z \Leftarrow (\text{let } x \Leftarrow M_1 \text{ in let } y \Leftarrow M_2 \text{ in } \langle x, y\rangle) \text{ in } (\pi_1 z)(\pi_2 z)) \\
&\quad = F^{\#}_{\mathbb{C}}(\Gamma \vdash \text{let } z \Leftarrow \langle M_1, M_2\rangle \text{ in } (\pi_1 z)(\pi_2 z)) \\
&\quad = F^{\#}_{\mathbb{C}}(z : (\tau_1 \to \tau_2) \times \tau_1 \vdash (\pi_1 z)(\pi_2 z) : \tau_2) \circ F^{\#}_{\mathbb{C}}(\Gamma \vdash \langle M_1, M_2\rangle) \\
&\quad = \mathrm{eval} \circ (\mathrm{id} \otimes F^{\#}_{\mathbb{C}}(\Gamma \vdash M_2 : \tau_1)) \circ (F^{\#}_{\mathbb{C}}(\Gamma \vdash M_1 : \tau_1 \to \tau_2) \otimes \mathrm{id}) \circ J\Delta.
\end{aligned}$$

MC6 has to hold by $(C5)$.

Hence $F^{\#}_{\mathbb{V}}$ and $F^{\#}_{\mathbb{C}}$ are indeed unique.

Hence the syntactic closed Freyd-category is indeed the free closed Freyd-category over a signature.

Note that this proves that λ_{C} is an internal language of Freyd-categories, and we can use it to prove statements about Freyd-categories the same way we can use the STLC to prove statements about CCCs.

5

The computational lambda calculus in the monadic metalanguage

This chapter describes and proves how to translate the computational lambda calculus to the monadic metalanguage in a semantically justified way.

By [22], [13], for $\mathcal{K}$ a cartesian category with a strong monad T and Kleisli-exponentials, $\mathcal{K} \xrightarrow{\eta\circ -} \mathcal{K}_T$ is a closed Freyd-category.

Chapter 3 characterized the syntactic CCC with a strong monad of the monadic metalanguage. Chapter 4 characterized how to interpret the computational lambda calculus in a Freyd-category. By interpreting the computational lambda calculus in the Freyd-category we get from the syntactic CCC with a strong monad of the monadic metalanguage, we get a mapping from computational lambda calculus terms to monadic metalanguage terms that is synthesized from purely semantic concerns. In particular, if $M_1 =_{\beta\eta} M_2$ in λ_{C}, then using the soundness result of λ_{C},

Computational lambda calculus $x : \tau \vdash M : \tau'$	Monadic metalanguage $x : \overline{\tau} \vdash \overline{M} : T\overline{\tau'}$
$x : \tau \vdash () : 1$	$x : \tau \vdash [()]_T : T1$
$x : \tau \vdash \pi_i M : \tau_i$	$x : \overline{\tau} \vdash \text{let } y \Leftarrow \overline{M} \text{ in } [\pi_i y]_T : T\overline{\tau_i}$
$x : \tau \vdash \langle M_1, M_2 \rangle : \tau_1 \times \tau_2$	$x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in } (\text{let } z_2 \Leftarrow \overline{M_2} \text{ in } \langle z_1, z_2 \rangle)$ $: T(\overline{\tau_1} \times \overline{\tau_2})$
$x : \tau \vdash \lambda z.M : \tau_1 \to \tau_2$	$x : \overline{\tau} \vdash [\lambda z.\overline{M}]_T : T(\overline{\tau_1} \to T\overline{\tau_2})$
$x : \tau \vdash M_1 M_2 : \tau_1 \times \tau_2$	$x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in } (\text{let } z_2 \Leftarrow \overline{M_2} \text{ in } z_1 z_2)$ $: T(\overline{\tau_1} \times \overline{\tau_2})$
$x : \tau \vdash \text{let } z \Leftarrow M_1 \text{ in } M_2 : \tau_2$	$x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } \overline{M_2} : T\overline{\tau_2}$

Figure 5.1: Translation of the computational lambda calculus to the monadic metalanguage

$[\![M_1]\!]_{\mathbb{C}} = [\![M_2]\!]_{\mathbb{C}}$ in the syntactic Freyd-category of λ_{ml}, so $\overline{M_1} =_{\beta\eta} \overline{M_2}$ in λ_{ml}.

The resulting translation is summarized recursively in Figure 5.1 and the derivation is described in Section 5.2.

5.1 Description of the Freyd-category structure derived from the monadic metalanguage

This section describes the the Freyd-category structure obtained from the syntactic CCC with a strong monad of the monadic metalanguage. It is derived by instantiating the structure of the syntactic CCC with a strong monad of the monadic metalanguage described in Chapter 3 in the standard construction of a Freyd-category from a CCC with a strong monad as in [22].

Finite products in $\mathcal{K}$:

- **Terminal object:** 1
- **Binary product of objects** τ_1**,** τ_2**:**

$$(\tau_1 \times \tau_2, (x : \tau_1 \times \tau_2 \vdash \pi_1 x : \tau_1), (x : \tau_1 \times \tau_2 \vdash \pi_2 x : \tau_2)).$$

Premonoidal structure in $\mathcal{K}_T$:

$$\tau_1 \otimes \tau_2 := \tau_1 \times \tau_2$$

For an object τ, for $(x : \tau_1 \vdash E : T\tau_2) \in \mathcal{K}(\tau_1, T\tau_2) = \mathcal{K}_T(\tau_1, \tau_2)$

$\tau \rtimes (x : \tau_1 \vdash E : T\tau_2) \in \mathcal{K}_T(\tau \times \tau_1, \tau \times \tau_2)$

$\tau \rtimes (x : \tau_1 \vdash E : T\tau_2) := (y : \tau \times \tau_1 \vdash \text{let } z \Leftarrow E[x \mapsto \pi_2 y] \text{ in } [\langle \pi_1 y, z\rangle]_T : T(\tau \times \tau_2))$

$(x : \tau_1 \vdash E : T\tau_2) \ltimes \tau \in \mathcal{K}_T(\tau_1 \times \tau, \tau_2 \times \tau)$

$(x : \tau_1 \vdash E : T\tau_2) \ltimes \tau := (y : \tau_1 \times \tau \vdash \text{let } z \Leftarrow E[x \mapsto \pi_1 y] \text{ in } [\langle z, \pi_2 y\rangle]_T : T(\tau_2 \times \tau))$

Closed structure in $\mathcal{K} \xrightarrow{\eta\circ -} \mathcal{K}_T$:

$$\tau_1 \Rightarrow \tau_2 := \tau_1 \to T\tau_2$$

$$\text{eval} \in \mathcal{K}_T((\tau_1 \Rightarrow \tau_2) \times \tau_1, \tau_2) = \mathcal{K}((\tau_1 \to T\tau_2) \times \tau_1, T\tau_2)$$

$$\text{eval} := x : (\tau_1 \to T\tau_2) \times \tau_1 \vdash (\pi_1 x)(\pi_2 x) : T\tau_2$$

$$\Lambda\left((x : \tau \times \tau_1 \vdash E : T\tau_2)\right) \in \mathcal{K}(\tau, \tau_1 \Rightarrow \tau_2)$$

$$\Lambda\left((x : \tau \times \tau_1 \vdash E : T\tau_2)\right) := y : \tau \vdash \lambda z : \tau_1.E : \tau_1 \to T\tau_2$$

5.2 Derivation of the translation

This section describes how the translation in Figure 5.1 is derived. In particular, to get the translation of a term M, which we denote by $\overline{M}$, we take its interpretation in the syntactic Freyd-category of the monadic metalanguage derived from its syntactic CCC with a strong monad and formulate it as a monadic metalanguage term.

Theorem 14. *Interpreting the computational lambda calculus in the syntactic Freyd-category of the monadic metalanguage synthesises the translation in Figure 5.1.*

Proof. For each of the term-constructors of the computational lambda calculus, we are going to confirm that the interpretation agrees with the monadic metalanguage term in Figure 5.1.

- **Case: unit**

$$\begin{aligned}
&x : \overline{\tau} \vdash \overline{()} : T\overline{1} \\
&\quad = [\![x : \tau \vdash () : 1]\!]_{\mathbb{C}} \\
&\quad = \eta \circ [\![x : \tau \vdash () : 1]\!]_{\mathbb{V}} \\
&\quad = \eta \circ !_{\overline{\tau}} \\
&\quad = (y : 1 \vdash [y]_T : T1) \circ (x : \overline{\tau} \vdash () : 1) \\
&\quad = (x : \overline{\tau} \vdash [()]_T : T1)
\end{aligned}$$

- **Case: proj**

$$
\begin{aligned}
& x : \overline{\tau} \vdash \overline{\pi_i M} : T\overline{\tau_i} \\
& \quad = [\![x : \tau \vdash \pi_i M : \tau_i]\!]_{\mathbb{C}} \\
& \quad = (\eta \circ \pi_i^{\mathcal{K}}) \circ_{\mathcal{K}_T} [\![x : \tau \vdash M : \tau_1 \times \tau_2]\!]_{\mathbb{C}} \\
& \quad = (\eta \circ (x : \overline{\tau_1} \times \overline{\tau_2} \vdash \pi_i x : \overline{\tau_i})) \circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \overline{M} : T\overline{\tau_1 \times \tau_2}) \\
& \quad = (y : \overline{\tau_1} \times \overline{\tau_2} \vdash [\pi_i y]_T : T\overline{\tau_i}) \circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \overline{M} : T\overline{\tau_1 \times \tau_2}) \\
& \quad = x : \overline{\tau} \vdash \text{let } y \Leftarrow \overline{M} \text{ in } [\pi_i y]_T : T\overline{\tau_i}
\end{aligned}
$$

- **Case: pair**

$$
\begin{aligned}
& x : \overline{\tau} \vdash \overline{\langle M_1, M_2 \rangle} : T\overline{\tau_1 \times \tau_2} \\
& \quad = [\![x : \tau \vdash \langle M_1, M_2 \rangle : \tau_1 \times \tau_2]\!]_{\mathbb{C}} \\
& \quad = (\overline{\tau_1} \rtimes [\![x : \tau \vdash M_2 : \tau_2]\!]_{\mathbb{C}}) \circ_{\mathcal{K}_T} ([\![x : \tau \vdash M_1 : \tau_1]\!]_{\mathbb{C}} \ltimes \overline{\tau}) \circ_{\mathcal{K}_T} (\eta \circ \Delta) \\
& \quad = (\overline{\tau_1} \rtimes (x : \tau \vdash \overline{M_2} : \overline{\tau_2})) \circ_{\mathcal{K}_T} ((x : \overline{\tau} \vdash \overline{M_1} : \overline{\tau_1}) \ltimes \overline{\tau}) \\
& \qquad\quad \circ_{\mathcal{K}_T} (x : \tau \vdash [\langle x, x \rangle]_T : T(\tau \times \tau)) \\
& \quad = (y_2 : \overline{\tau_1} \times \overline{\tau} \vdash \text{let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 y_2] \text{ in } [\langle \pi_1 y_2, z_2 \rangle]_T : T(\overline{\tau_1} \times \overline{\tau_2})) \\
& \qquad\quad \circ_{\mathcal{K}_T} (y_1 : \overline{\tau} \times \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1}[x \mapsto \pi_1 y_1] \text{ in } [\langle z_1, \pi_2 y_1 \rangle]_T : T(\overline{\tau_1} \times \overline{\tau})) \\
& \qquad\quad \circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash [\langle x, x \rangle]_T : T(\overline{\tau} \times \overline{\tau})) \\
& \quad = (y_2 : \overline{\tau_1} \times \overline{\tau} \vdash \text{let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 y_2] \text{ in } [\langle \pi_1 y_2, z_2 \rangle]_T : T(\overline{\tau_1} \times \overline{\tau_2})) \\
& \qquad\quad \circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \text{let } y_1 \Leftarrow [\langle x, x \rangle]_T \text{ in } (\text{let } z_1 \Leftarrow \overline{M_1}[x \mapsto \pi_1 y_1] \\
& \qquad\qquad\quad \text{in } [\langle z_1, \pi_2 y_1 \rangle]_T) : T(\overline{\tau_1} \times \overline{\tau})) \\
& \quad \overset{\text{l}\beta}{=} (y_2 : \overline{\tau_1} \times \overline{\tau} \vdash \text{let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 y_2] \text{ in } [\langle \pi_1 y_2, z_2 \rangle]_T : T(\overline{\tau_1} \times \overline{\tau_2})) \\
& \qquad\quad \circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1}[x \mapsto \pi_1 \langle x, x \rangle] \text{ in } [\langle z_1, \pi_2 \langle x, x \rangle \rangle]_T : T(\overline{\tau_1} \times \overline{\tau})) \\
& \quad \overset{\text{p}\beta}{=} (y_2 : \overline{\tau_1} \times \overline{\tau} \vdash \text{let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 y_2] \text{ in } [\langle \pi_1 y_2, z_2 \rangle]_T : T(\overline{\tau_1} \times \overline{\tau_2}))
\end{aligned}
$$

$$
\begin{aligned}
& \circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in } [\langle z_1, x\rangle]_T : T(\overline{\tau_1} \times \overline{\tau})) \\
&= x : \overline{\tau} \vdash \text{let } y_2 \Leftarrow (\text{let } z_1 \Leftarrow \overline{M_1} \text{ in } [\langle z_1, x\rangle]_T) \\
&\qquad \text{in let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 y_2] \text{ in } [\langle \pi_1 y_2, z_2\rangle]_T : T(\overline{\tau_1} \times \overline{\tau_2}) \\
&\overset{\text{a}}{=} x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in } (\text{let } y_2 \Leftarrow [\langle z_1, x\rangle]_T \\
&\qquad \text{in } (\text{let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 y_2] \text{ in } [\langle \pi_1 y_2, z_2\rangle]_T)) \\
&\overset{\text{l}\beta}{=} x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in } (\text{let } z_2 \Leftarrow \overline{M_2}[x \mapsto \pi_2 [\langle z_1, x\rangle]_T] \\
&\qquad \text{in } [\langle \pi_1 [\langle z_1, x\rangle]_T, z_2\rangle]_T) \\
&\overset{\text{p}\beta}{=} x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in } (\text{let } z_2 \Leftarrow \overline{M_2} \text{ in } [\langle z_1, z_2\rangle]_T) : T(\overline{\tau_1} \times \overline{\tau_2})
\end{aligned}
$$

- **Case: abst**

$$
\begin{aligned}
& x : \overline{\tau} \vdash \overline{\lambda z.M} : T\overline{\tau_1 \to \tau_2} \\
&\quad = [\![x : \tau \vdash \lambda z.M : \tau_1 \to \tau_2]\!]_{\mathbb{C}} \\
&\quad = \eta \circ [\![x : \tau \vdash \lambda z.M : \tau_1 \to \tau_2]\!]_{\mathbb{V}} \\
&\quad = \eta \circ \Lambda([\![x : \tau, z : \tau_1 \vdash M : \tau_2]\!]_{\mathbb{C}}) \\
&\quad = \eta \circ \Lambda([\![y : \tau \times \tau_1 \vdash M[x \mapsto \pi_1 y, z \mapsto \pi_2 y] : \tau_2]\!]_{\mathbb{C}}) \\
&\quad = \eta \circ \Lambda(y : \overline{\tau \times \tau_1} \vdash \overline{M}[x \mapsto \pi_1 y, z \mapsto \pi_2 y] : \overline{\tau_2}) \\
&\quad = \eta \circ \Lambda(y : \overline{\tau} \times \overline{\tau_1} \vdash \overline{M}[x \mapsto \pi_1 y, z \mapsto \pi_2 y] : \overline{\tau_2}) \\
&\quad = \eta \circ \Lambda(x : \overline{\tau}, z : \overline{\tau_1} \vdash \overline{M} : T\overline{\tau_2}) \\
&\quad = \eta \circ (y : \overline{\tau} \vdash \lambda z.\overline{M} : \overline{\tau_1} \to T\overline{\tau_2}) \\
&\quad = (y : \overline{\tau} \vdash [\lambda z.\overline{M}]_T : T(\overline{\tau_1} \to T\overline{\tau_2}))
\end{aligned}
$$

- **Case: app**

$$x : \overline{\tau} \vdash \overline{M_1 M_2} : T\overline{\tau_2}$$

$$= [\![x : \tau \vdash M_1 M_2 : \tau_2]\!]_{\mathbb{C}}$$

$$= \text{eval} \circ_{\mathcal{K}_T} [\![x : \tau \vdash \langle M_1, M_2 \rangle : (\tau_1 \Rightarrow \tau_2) \times \tau_1]\!]_{\mathbb{C}}$$

$$= (y : (\overline{\tau_1 \Rightarrow \tau_2}) \times \overline{\tau_1} \vdash (\pi_1 y)(\pi_2 y) : T\overline{\tau_2})$$

$$\circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \overline{\langle M_1, M_2 \rangle} : T((\overline{\tau_1 \Rightarrow \tau_2) \times \tau_1}))$$

$$= (y : (\overline{\tau_1 \Rightarrow \tau_2}) \times \overline{\tau_1} \vdash (\pi_1 y)(\pi_2 y) : T\overline{\tau_2})$$

$$\circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1}$$

$$\text{in (let } z_2 \Leftarrow \overline{M_2} \text{ in } \langle z_1, z_2 \rangle) : T((\overline{\tau_1 \Rightarrow \tau_2) \times \tau_1}))$$

$$= x : \overline{\tau} \vdash \text{let } y \Leftarrow (\text{let } z_1 \Leftarrow \overline{M_1} \text{ in (let } z_2 \Leftarrow \overline{M_2} \text{ in } \langle z_1, z_2 \rangle))$$

$$\text{in } (\pi_1 y)(\pi_2 y) : T\overline{\tau_2}$$

$$\stackrel{\text{a}}{=} x : \overline{\tau} \vdash$$

$$\text{let } z_1 \Leftarrow \overline{M_1} \text{ in let } y \Leftarrow (\text{let } z_2 \Leftarrow \overline{M_2} \text{ in } \langle z_1, z_2 \rangle)$$

$$\text{in } (\pi_1 y)(\pi_2 y) : T\overline{\tau_2}$$

$$\stackrel{\text{a}}{=} x : \overline{\tau} \vdash$$

$$\text{let } z_1 \Leftarrow \overline{M_1} \text{ in (let } z_2 \Leftarrow \overline{M_2} \text{ in (let } y \Leftarrow \langle z_1, z_2 \rangle \text{ in } (\pi_1 y)(\pi_2 y))) : T\overline{\tau_2}$$

$$= x : \overline{\tau} \vdash \text{let } z_1 \Leftarrow \overline{M_1} \text{ in (let } z_2 \Leftarrow \overline{M_2} \text{ in } z_1 z_2) : T\overline{\tau_2}$$

- **Case: let**

$$x : \overline{\tau} \vdash \overline{\text{let } z \Leftarrow M_1 \text{ in } M_1} : T\overline{\tau_2}$$

$$= [\![x : \tau \vdash \text{let } z \Leftarrow M_1 \text{ in } M_1 : \tau_2]\!]_{\mathbb{C}}$$

$$= [\![x : \tau, z : \tau_1 \vdash M_2 : \tau_2]\!]_{\mathbb{C}} \circ_{\mathcal{K}_T} ([\![\tau]\!] \rtimes [\![x : \tau \vdash M_1 : \tau_1]\!]_{\mathbb{C}}) \circ_{\mathcal{K}_T} (\eta \circ \Delta)$$

$$= (x : \overline{\tau}, z : \overline{\tau_1} \vdash \overline{M_2} : T\overline{\tau_2}) \circ_{\mathcal{K}_T} (\overline{\tau} \rtimes (x : \overline{\tau} \vdash \overline{M_1} : T\overline{\tau_1}))$$

$$\circ_{\mathcal{K}_T} (\eta \circ (x : \overline{\tau} \vdash \langle x, x \rangle : \overline{\tau}))$$

$$= (x : \overline{\tau}, z : \overline{\tau_1} \vdash \overline{M_2} : T\overline{\tau_2})$$

$$
\begin{aligned}
&\circ_{\mathcal{K}_T} (y : \overline{\tau} \times \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1}[x \mapsto \pi_2 y] \text{ in } [\pi_1 y]_T z : \overline{\tau} \times \overline{\tau_1}) \\
&\circ_{\mathcal{K}_T} (\eta \circ (x : \overline{\tau} \vdash \langle x, x\rangle : \overline{\tau})) \\
= {}& (x : \overline{\tau}, z : \overline{\tau_1} \vdash \overline{M_2} : T\overline{\tau_2}) \\
&\circ_{\mathcal{K}_T} (y : \overline{\tau} \times \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1}[x \mapsto \pi_2 y] \text{ in } [\langle \pi_1 y, z\rangle]_T : \overline{\tau} \times \overline{\tau_1}) \\
&\circ (x : \overline{\tau} \vdash \langle x, x\rangle : \overline{\tau}) \\
= {}& (x : \overline{\tau}, z : \overline{\tau_1} \vdash \overline{M_2} : T\overline{\tau_2}) \\
&\circ_{\mathcal{K}_T} (x : \overline{\tau} : \overline{\tau} \times \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1}[x \mapsto \pi_2 \langle x, x\rangle] \\
&\qquad \text{in } [\langle \pi_1 \langle x, x\rangle, z\rangle]_T : \overline{\tau} \times \overline{\tau_1}) \\
= {}& (x : \overline{\tau}, z : \overline{\tau_1} \vdash \overline{M_2} : T\overline{\tau_2}) \\
&\circ_{\mathcal{K}_T} (x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } [\langle x, z\rangle]_T : \overline{\tau} \times \overline{\tau_1}) \\
= {}& x : \overline{\tau} \vdash \text{let } y \Leftarrow (\text{let } z \Leftarrow \overline{M_1} \text{ in } [\langle x, z\rangle]_T) \text{ in } \overline{M_2}[x \mapsto \pi_1 y, z \mapsto \pi_2 y] : T\overline{\tau_2} \\
\stackrel{\text{a}}{=} {}& x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } (\text{let } y \Leftarrow [\langle x, z\rangle]_T \text{ in } \overline{M_2}[x \mapsto \pi_1 y, z \mapsto \pi_2 y]) : T\overline{\tau_2} \\
\stackrel{\text{l}\beta}{=} {}& x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } \overline{M_2}[x \mapsto \pi_1 \langle x, z\rangle, z \mapsto \pi_2 \langle x, z\rangle] \\
\stackrel{\text{p}\beta}{=} {}& x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } (\text{let } y \Leftarrow [\langle x, z\rangle]_T \text{ in } \overline{M_2}[x \mapsto \pi_1 y, z \mapsto \pi_2 y]) : T\overline{\tau_2} \\
\stackrel{\text{l}\beta}{=} {}& x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } \overline{M_2}[x \mapsto x, z \mapsto z] : T\overline{\tau_2} \\
= {}& x : \overline{\tau} \vdash \text{let } z \Leftarrow \overline{M_1} \text{ in } \overline{M_2} : T\overline{\tau_2}
\end{aligned}
$$

Hence the interpretation indeed synthesises the above translation. □

6
Conclusion

This dissertation studied the category-theoretic semantics of simply-typed programming languages. It surveyed some of the key results relating to the simply-typed lambda calculus and the monadic metalanguage and their category-theoretic semantics. It then formalized the corresponding result relating to the computational lambda calculus and Freyd-categories. Finally, it used these semantics to give a semantically-justified translation from the computational lambda calculus to the monadic metalanguage.

While it has been known that Freyd-categories provide a sound and complete semantics of the computational lambda calculus, this is the first full description and proof of the denotational semantics directly in Freyd-categories and the first derivation of the translation of the computational lambda calculus to the monadic metalanguage using it.

Directions for future work could include formalizing further translations between other pairs of languages based on their semantics, such as a translation between the fine-grain call-by-value and the computational lambda calculus. Another direction could be showing that the semantics given indirectly by translating to another language first, such as in [13], is indeed equivalent to the one directly given in this dissertation. Additionally, other aspects found in real languages, such as sum types, or more ambitiously, object-oriented features could be explored.

A

Notation

$\Diamond$	An empty context.
Δ	In a cartesian category $\mathcal{C}$, for an object X, a morphism $\Delta : X \to X$ given by $\Delta = \langle \mathrm{id}_X, \mathrm{id}_X \rangle$.
$\stackrel{\text{IH}}{=}$	Holds by the inductive hypothesis.
$\stackrel{\text{w}}{=}$	Holds by the weakening lemma.
$\stackrel{\text{s}}{=}$	Holds by the substitution lemma.
$\stackrel{\text{l}\beta}{=}$	Holds by letβ.
$\stackrel{\text{l}\eta}{=}$	Holds by letη.
$\stackrel{\text{p}\beta}{=}$	Holds by prodβ.
$\stackrel{\text{p}\eta}{=}$	Holds by prodη.
$\stackrel{\text{f}\beta}{=}$	Holds by fnβ.
$\stackrel{\text{f}\eta}{=}$	Holds by fnη.
$\stackrel{\text{cp}}{=}$	Holds by comppair.
$\stackrel{\text{ca}}{=}$	Holds by compapp.
$\stackrel{\text{cpr}}{=}$	Holds by compproj.
$\stackrel{\text{a}}{=}$	Holds by assoc.
$\stackrel{\text{u}}{=}$	Holds by unit.

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