Logical relations for CBPV models, via internal fibrations in a 2-category

LICS 2025, Singapore

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What’s it all about?

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the first complete denotational
account of logical relations for CBPV

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2025 Alonzo Church prize winner!

Levy’s call-by-push-value

= an effectful language with fine control over when effects happen

subsumes both CBV and CBN

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the first complete denotational account of logical relations for CBPV

a classic proof technique for logics and programming languages

"operational": termination, normalisation, …

"denotational": definability, simulation, …

= characterising the definable morphisms in a model

= how is $[\![P]\!]^{\mathscr{M}}$ related to $[\![P]\!]^{\mathscr{N}}$?

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logical relations as a way to build denotational models

related work:
Kammar (2014), McDermott (2020), …

other approaches:
Levy (2012), Goncharov—Tsampas—Urbat (2025), …

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What you get

1. Principled definition of relations models for CBPV
2. Lots of applications and examples, via a lifting theorem
   eg. effect simulation, $\top\top$-lifting, syntactic definition, Lafont argument, …
3. General theory encompassing STLC, λ_{ml} and CBPV

simply-typed λ-calculus

monadic metalanguage

Logical relations (in their simplest form) Milner, Plotkin, ….

A logical relation R consists of:

a predicate $R_A \subseteq [\![A]\!]$ for each type A…

…determined inductively at higher types by a logical relations condition

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Eg. logical relations condition for $\to$ in STLC:

$$f \in R_{A \to B} \subseteq [\![A]\!] \Rightarrow [\![B]\!] \qquad \text{iff} \qquad f(x) \in R_B \text{ whenever } x \in R_A$$

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Basic Lemma:

every program is in the relation: $[\![P]\!] \in R_A$ for any closed $P : A$

so long as this holds for the basic constants

Logical relations, denotationally

Statman,
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semantic model

Set

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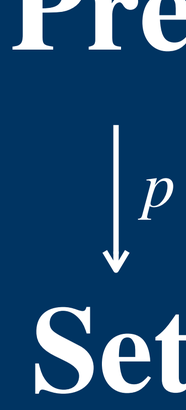
objects: $(X \in \mathbf{Set}, R \subseteq X)$
maps: functions preserving the predicate



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Pred is cartesian closed:

$$(X, R) \Rightarrow (Y, S) := (X \Rightarrow Y, R \supset S)$$

$f \in R \supset S$ iff $f(x) \in S$ whenever $x \in R$

$f \in [\![A \to B]\!]^{\mathbf{Pred}}$ iff f preserves the predicate on $[\![A]\!]^{\mathbf{Set}}$

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forgetful functor p
strictly preserves CC-structure

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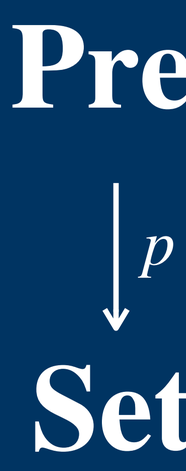

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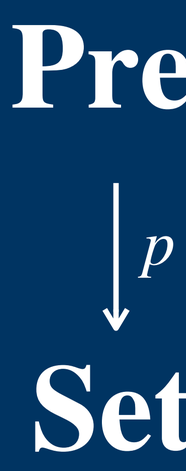

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Basic Lemma holds automatically:

$$p\left([\![P]\!]^{\mathbf{Pred}}\right) = [\![P]\!]^{\mathbf{Set}}$$

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Logical relations = relations models

Hermida, Jacobs, Katsumata, ….

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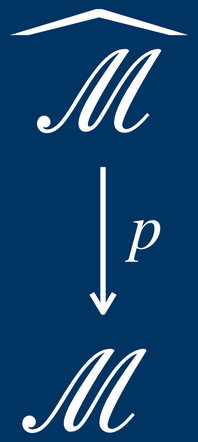

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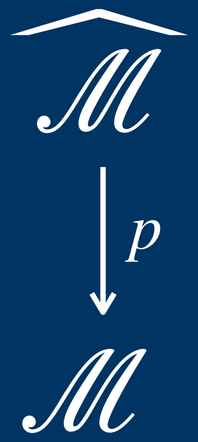

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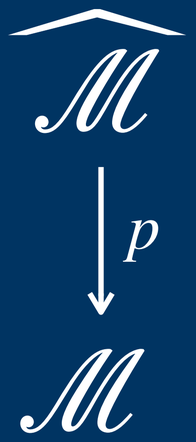

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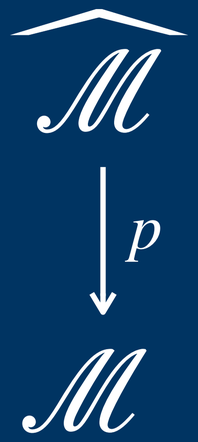

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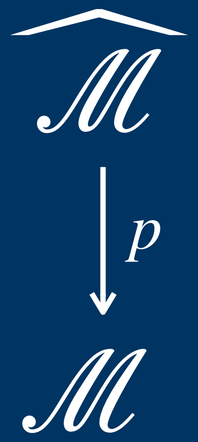

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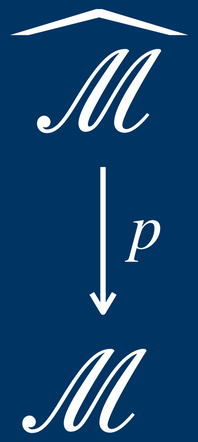

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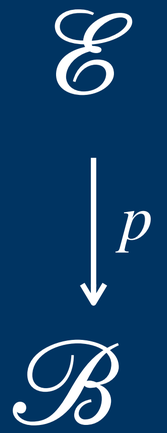

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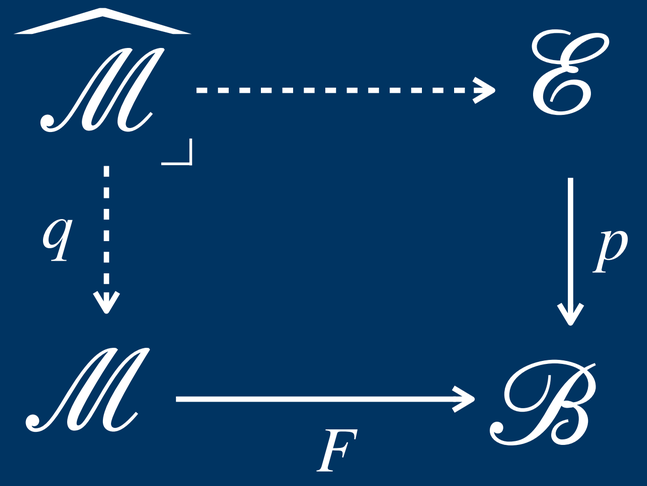

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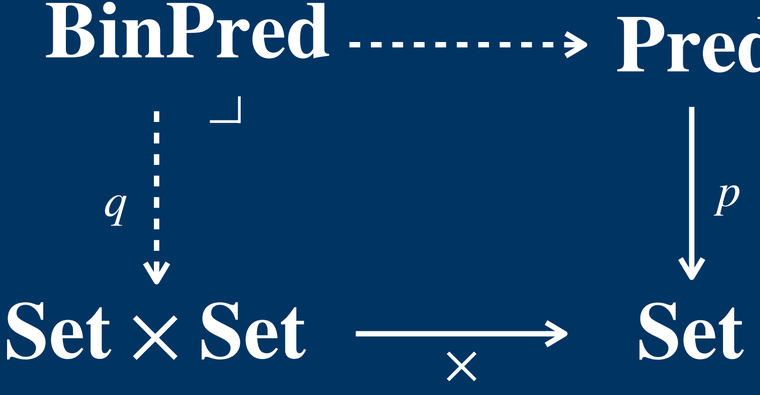
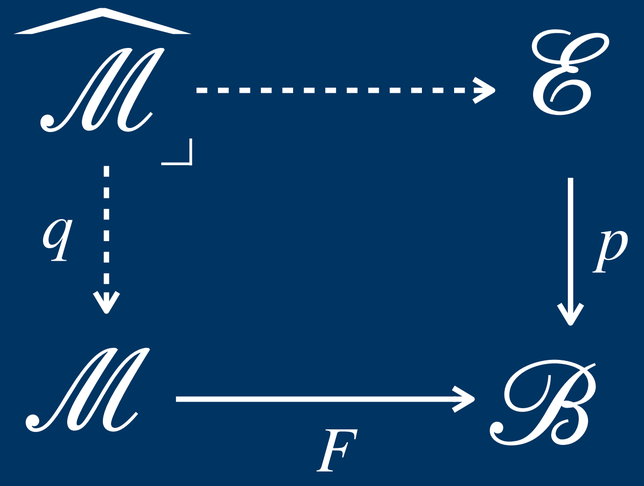
objects: $(X \in \mathbf{Set}, R \subseteq X \times X)$
maps: pairs of maps preserving the relation





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simply-typed
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This works for STLC and λ_{ml} and λ_{c}

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eg. Katsumata, McDermott—Kammar, Goubault-Larrecq et al, …

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Why?

semantic models are more subtle:
uses enriched categories

want to say ‘fibration preserving model structure’,
but what is the right notion of fibration?

What we do:

turn to 2-category theory

= a language for structures on categories

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Construct a 2-category of CBPV models

→ fibration for logical relations := fibrations internal to this

→ key examples from the general theory eg. subobject, codomain, …

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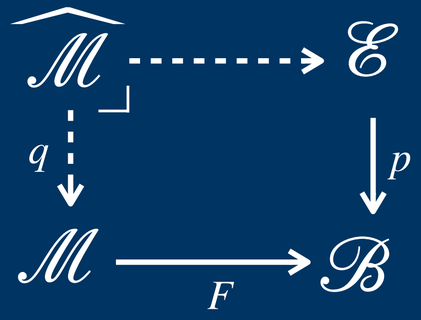

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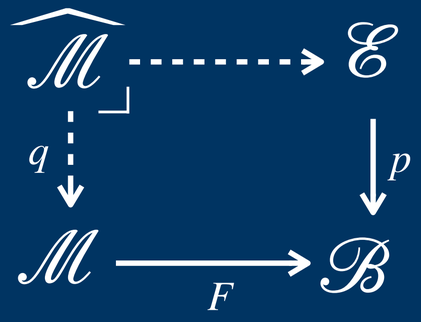
Get many new CBPV models

eg. effect simulation, ⊤⊤-lifting, syntactic defn, conservativity,…



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email me: p.saville@sussex.ac.uk
arxiv link: 10.48550/arXiv.2505.14482
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