A 2-categorical approach to logical relations

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These slides available at philipsaville.co.uk

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Things I learned from Marcelo

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1. Look for the right abstractions.

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1. Look for the right **abstractions**.
2. Let the algebra tell you **how things should be**.

"correct-by-construction"

"principled"

"synthesise"

Logical relations, at varying levels of abstraction

Level 1: logical relations as families of relations Milner, Plotkin, ….

A logical relation R consists of:

a predicate $R_A \subseteq [\![A]\!]$ for each type A…

…determined inductively at higher types by a logical relations condition

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E.g. logical relations condition for $\rightarrow$ in STLC:

$$f \in R_{A \rightarrow B} \subseteq [\![A]\!] \Rightarrow [\![B]\!] \quad \text{iff} \quad f(x) \in R_B \text{ whenever } x \in R_A$$

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A logical relation R consists of:

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Basic Lemma:

every program is in the relation: $[\![P]\!] \in R_A$ for any closed $P : A$

so long as this holds for the basic constants

Level 2: logical relations as relations models

Statman,
Reynolds & Ma, ….

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semantic model

Set

Level 2: logical relations as relations models

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objects: $(X \in \mathbf{Set}, R \subseteq X)$
maps: functions preserving the predicate

$$\begin{array}{c}\mathbf{Pred}\\ \downarrow p\\ \mathbf{Set}\end{array}$$

semantic model

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$$\mathbf{Pred}$$
$$\downarrow p$$
$$\mathbf{Set}$$

semantic model

$\mathbf{Pred}$ is cartesian closed:

$$(X, R) \Rightarrow (Y, S) := (X \Rightarrow Y, R \supset S)$$

$f \in R \supset S$ iff $f(x) \in S$ whenever $x \in R$

$f \in [\![A \to B]\!]^{\mathbf{Pred}}$ iff f preserves the predicate on $[\![A]\!]^{\mathbf{Set}}$

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CC-structure encodes the logical relations condition!

Basic Lemma holds automatically:

$$p\left([\![P]\!]^{\mathbf{Pred}}\right) = [\![P]\!]^{\mathbf{Set}}$$

Level 3: logical relations as certain fibrations

Hermida, Jacobs, Katsumata, …

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semantic model $\mathscr{M}$

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‘relations model’	$\widehat{\mathscr{M}}$
forgetful functor p strictly preserves model-structure	$\downarrow p$
semantic model	$\mathscr{M}$

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fibration for logical relations:
fibration
strictly preserving
model structure

Logical relations can be seen as…

Level 1: type-indexed families of relations

Level 2: relations models

Level 3: fibrations for logical relations

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Logical relations can be seen as…

Level 1: type-indexed families of relations

Level 2: relations models

Level 3: fibrations for logical relations

also get:
a good theory for building
new logical relations

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Building fibrations for logical relations

e.g. effect simulation, ⊤⊤-lifting, Kripke relations of varying arity, …

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$$\begin{array}{c}\mathscr{E}\\ \downarrow p\\ \mathscr{B}\end{array}$$

abstract notion of 'relation'

ie. p a fibration for logical relations

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$$\begin{array}{ccc} & & \mathscr{E} \\ & & \downarrow{\scriptstyle p} \\ \mathscr{M} & \xrightarrow[F]{} & \mathscr{B} \end{array}$$

semantic model

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weak map of models

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lifting theorem:

universal choice of model

'glueing' objects of $\mathscr{M}$ to 'relations' in $\mathscr{E}$

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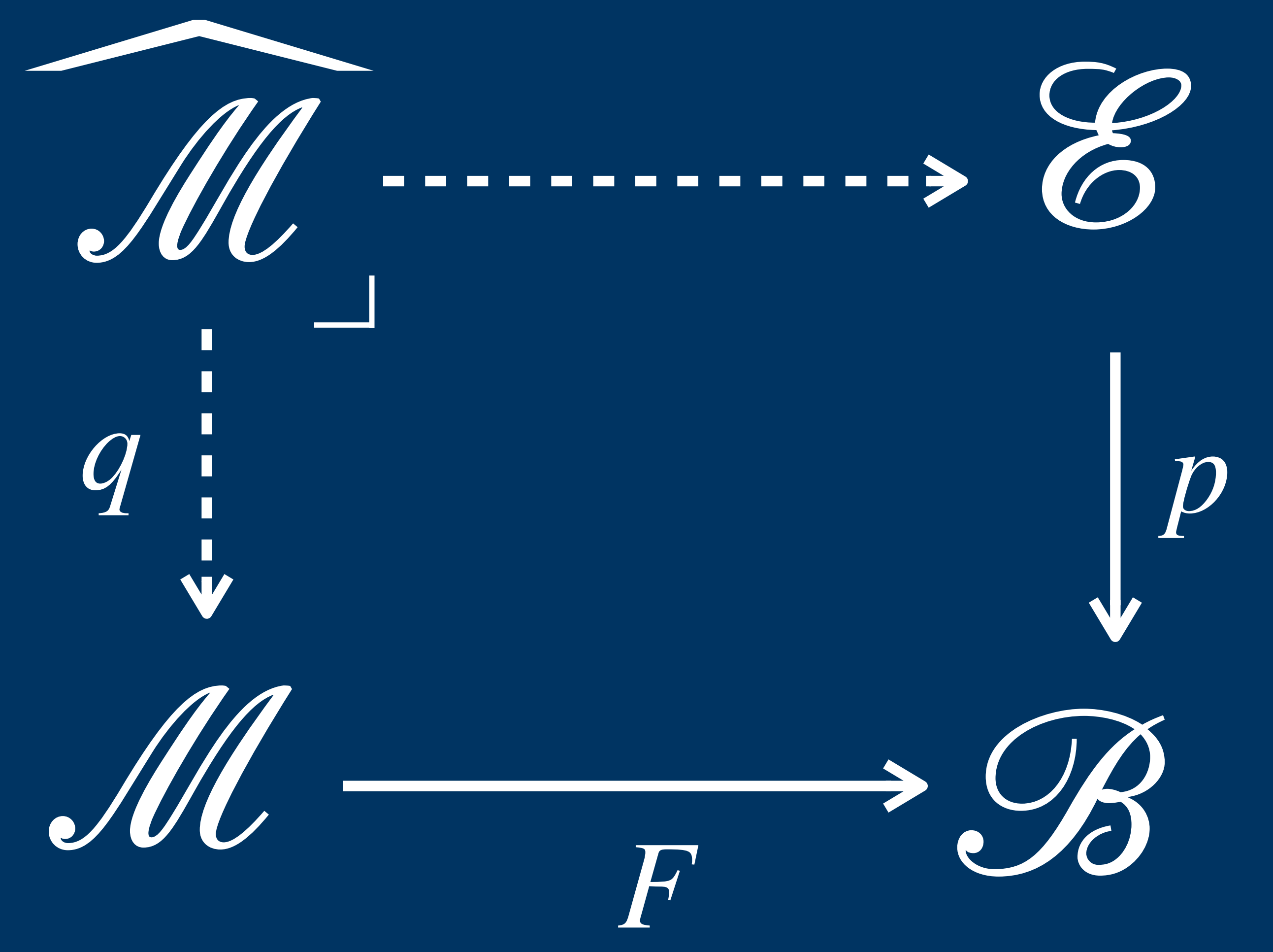

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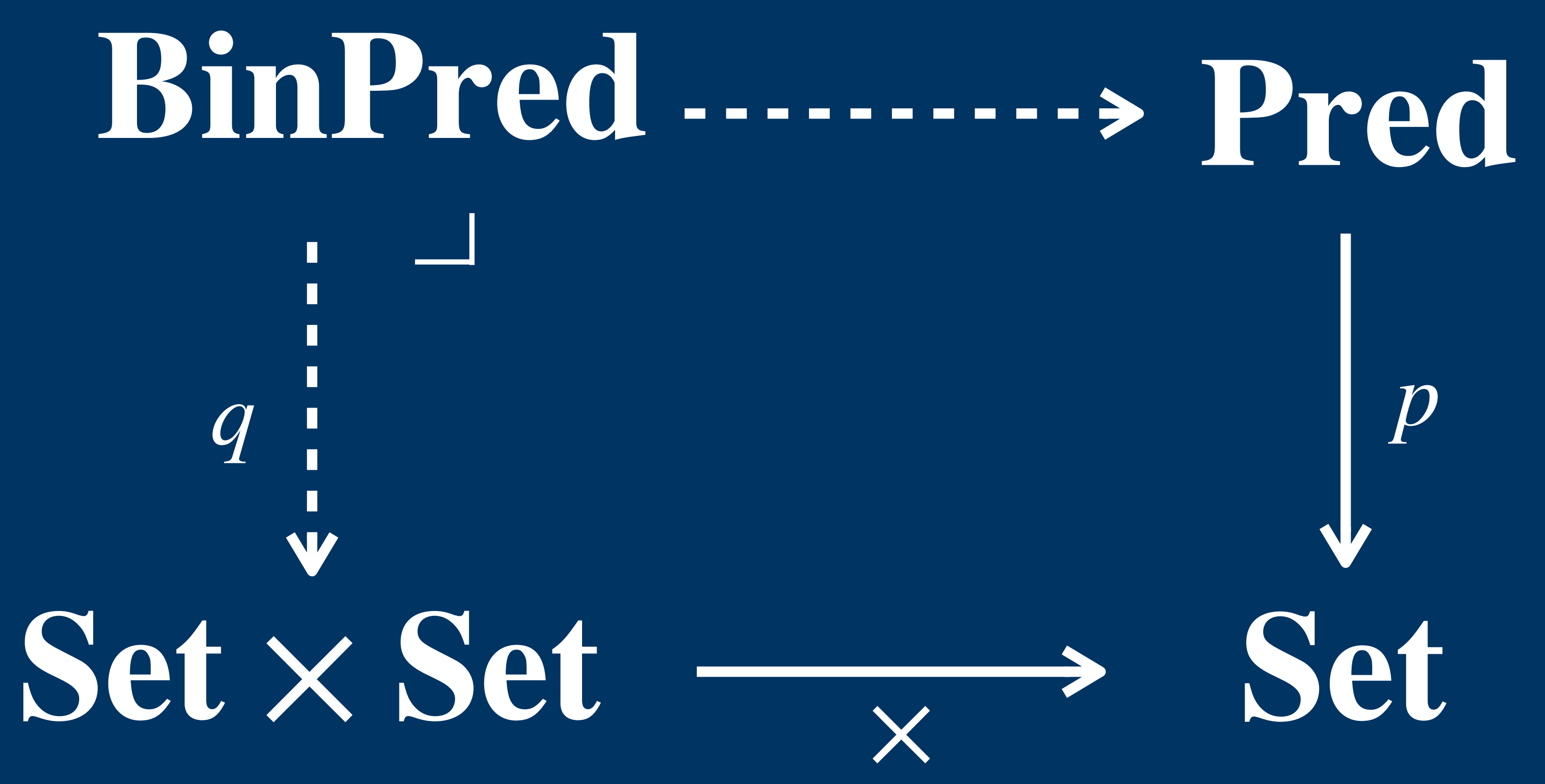
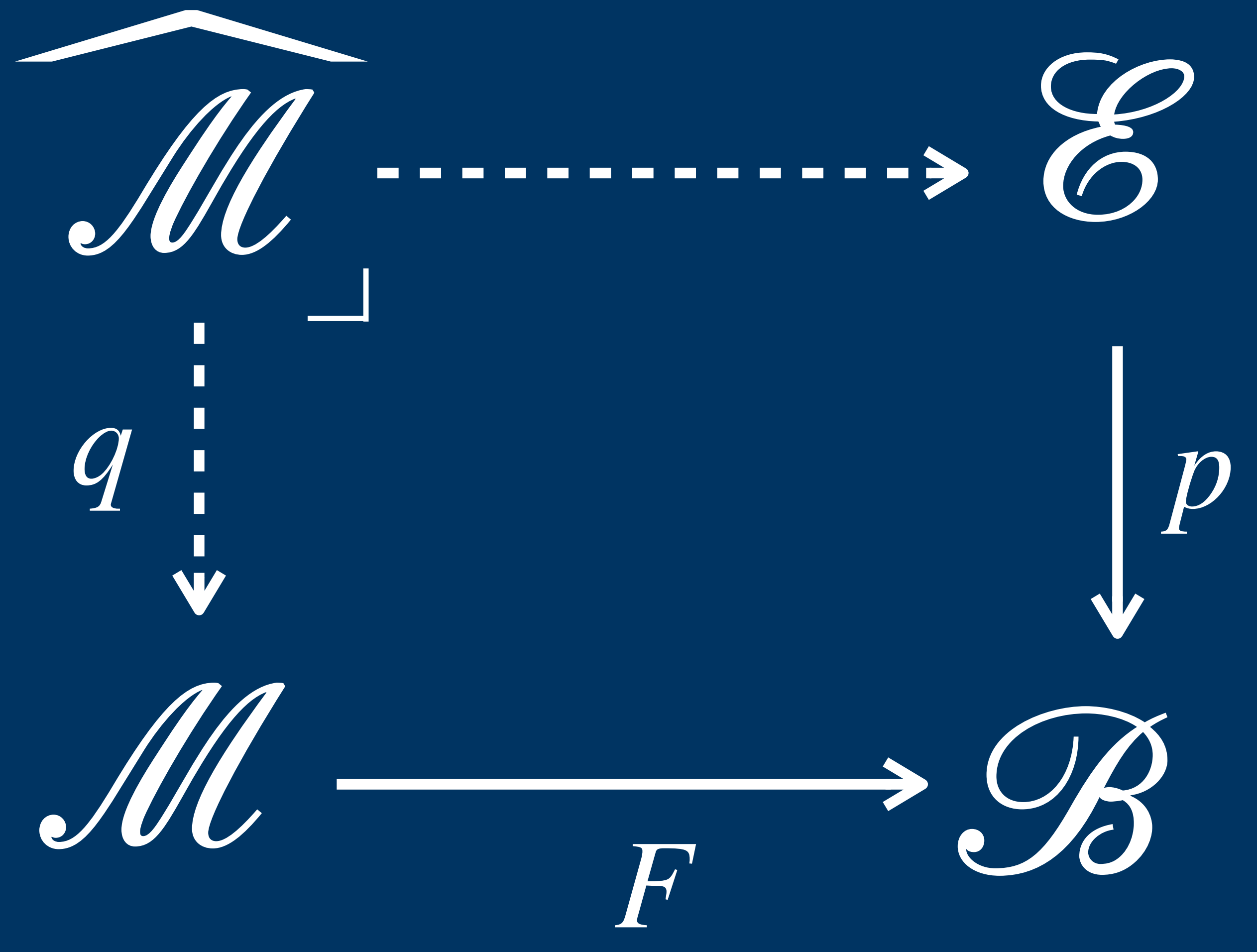
objects: $(X \in \mathbf{Set}, R \subseteq X \times X)$
maps: pairs of maps preserving the relation





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Logical relations can be seen as…

Level 1: type-indexed families of relations

Level 2: relations models

Level 3: fibrations for logical relations

+ can build new examples
from old ones by glueing

fibration for logical relations:

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Logical relations: a 2-categorical perspective

What's a fibration for logical relations for call-by-push-value?

A CBPV-model is a $\mathrm{Psh}(\mathscr{V})$-enriched adjunction

$$\mathrm{self}\,\mathscr{V} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{C} \quad (F \dashv U)$$

1. What is a CBPV fibration for logical relations?
2. Do they satisfy a lifting theorem?

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1. What is a CBPV fibration for logical relations?
2. Do they satisfy a lifting theorem?

can no longer just say a fibration which preserves all the model structure

need a sensible notion of glueing

Observation:

“fibration which preserves model structure”	$\approx$	fibration internal to 2-category of models

A fibration internal to… is…

$\mathbf{CartCart}_{\text{strict}}$	a fibration strictly preserving products
$\mathbf{CCCart}_{\text{strict}}$	a fibration strictly preserving CC-structure
λ_{ml}-models	a fibration strictly preserving λ_{ml}-structure

Suggestion for level 4:

fibration for logical relations	:=	fibration internal to 2-category of models

*sometimes sufficient but not necessary condition, but a lot of the theory still applies

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Does this work correctly?

Internal fibrations are always fibrations which preserve model structure

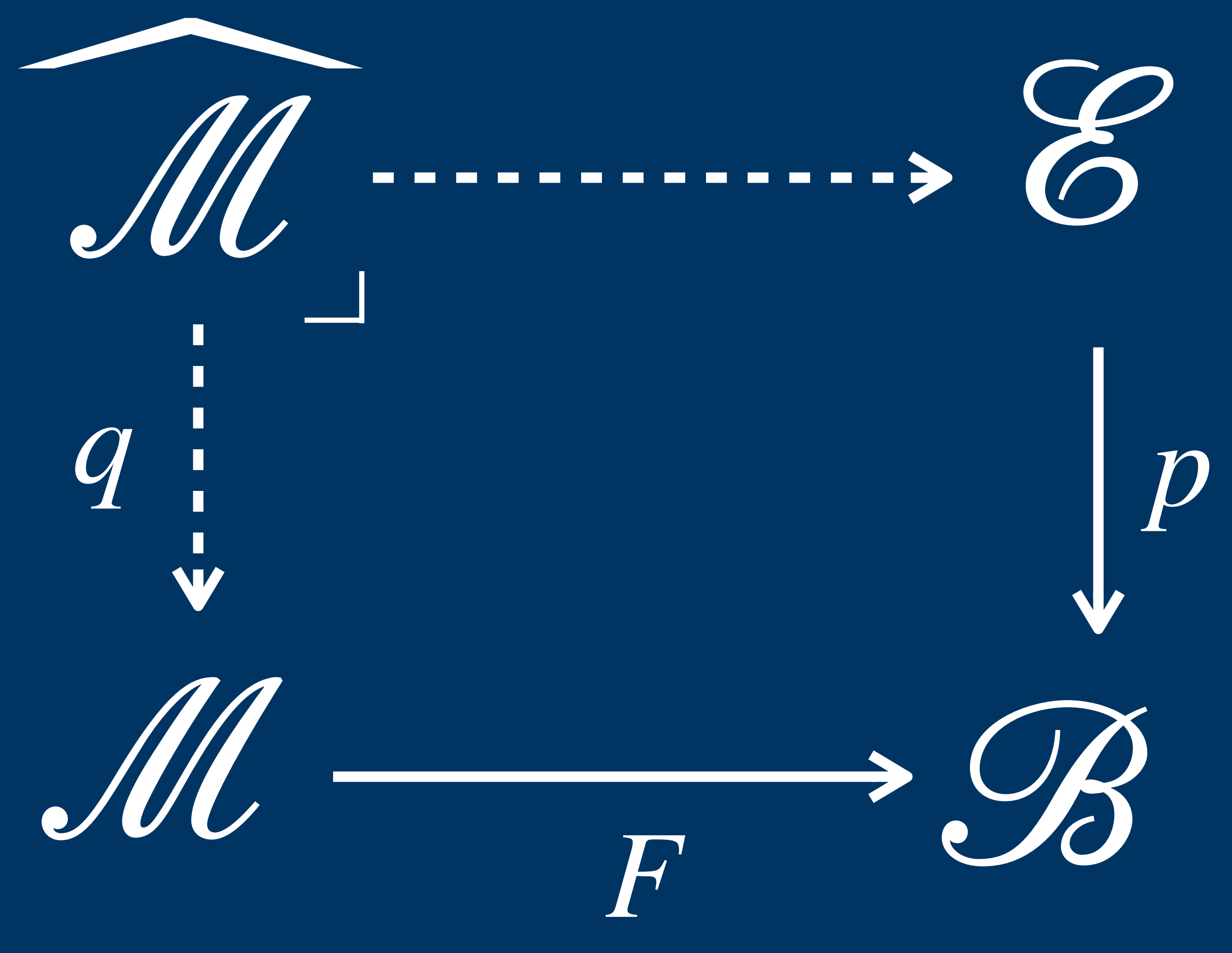
Theorem: let T be a relative pseudomonad along a pseudofunctor $J : \mathscr{A} \to \mathscr{C}$. Then a fibration in $\mathrm{Ps}\text{-}T\text{-}\mathrm{Alg}_{\mathrm{ps}}$ is exactly an algebra pseudomorphism $(p, \overline{p})$ such that p is a fibration in $\mathscr{C}$.

Glueing plays well with fibrations

lifting theorem:

universal choice of model
'glueing' objects of $\mathscr{M}$ to 'relations' in $\mathscr{E}$

semantic model



abstract notion of 'relation'

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weak map of models

Proposition [classic]: fibrations are closed under pullback.

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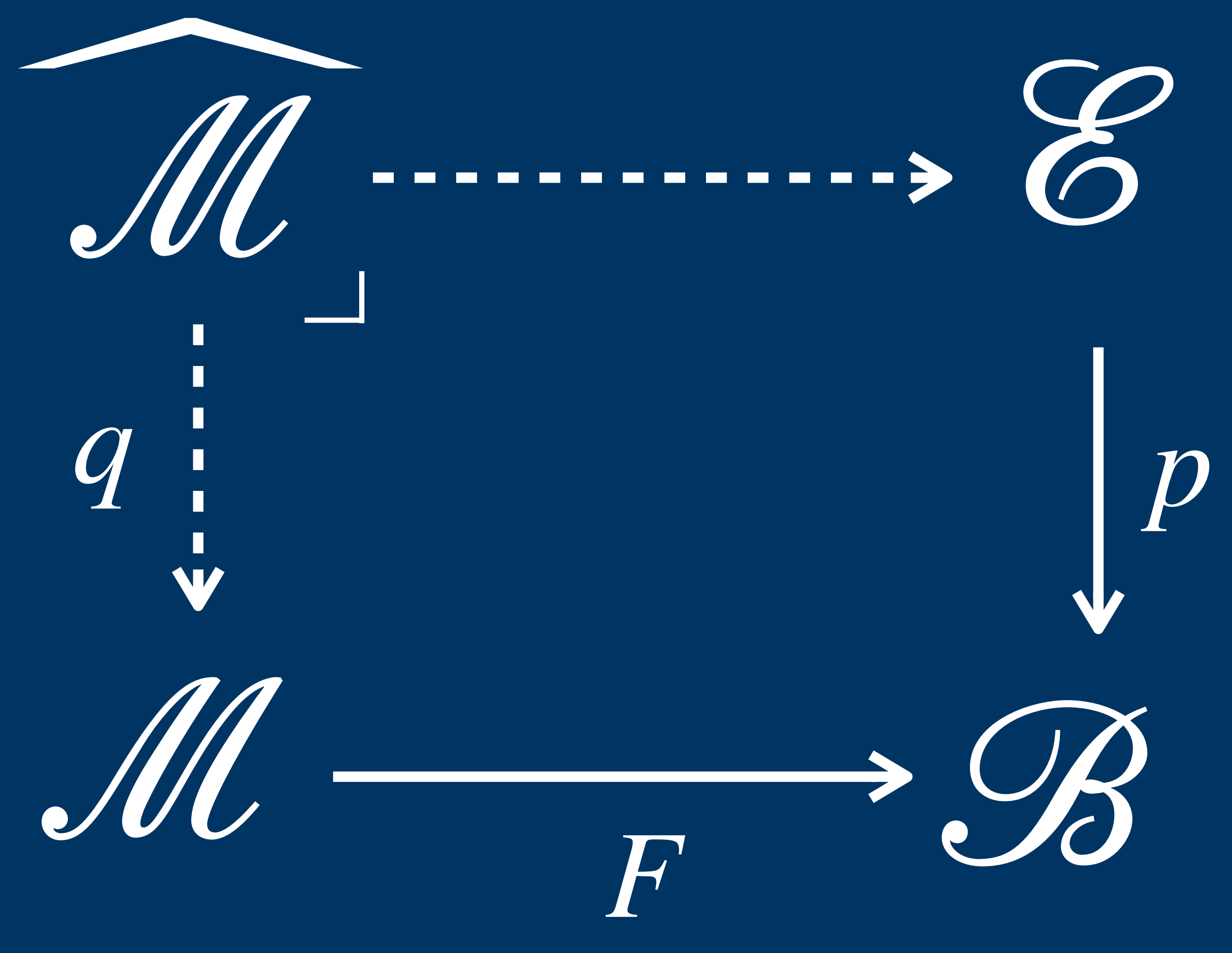

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Proposition [classic]: fibrations are closed under pullback.

Glueing reduces to the existence of certain pullbacks in
2-categories of models.

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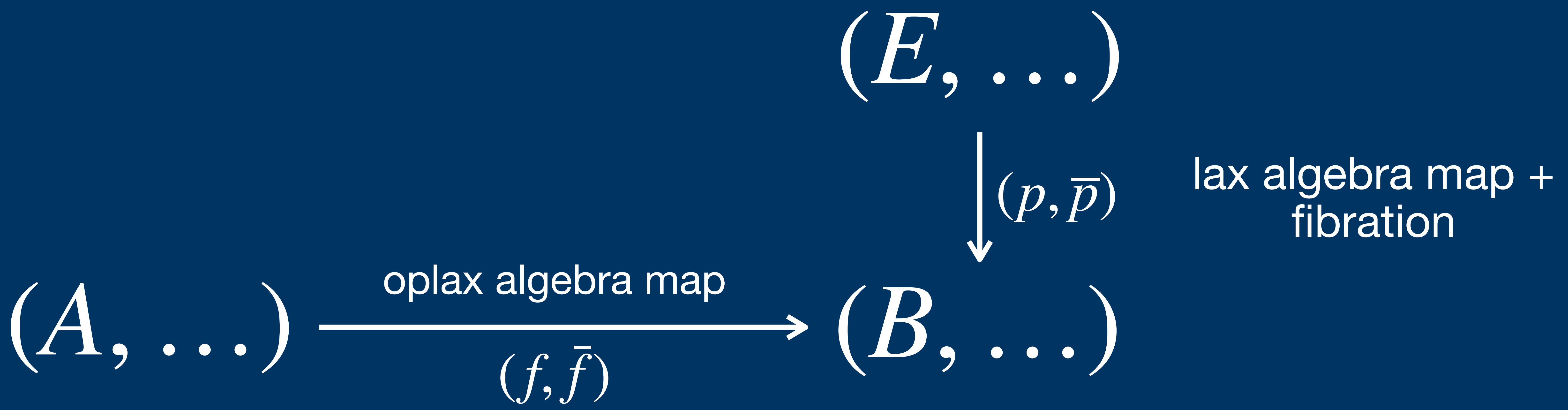
Proposition: let T be a relative pseudomonad. Consider the following cospan of pseudoalgebras:



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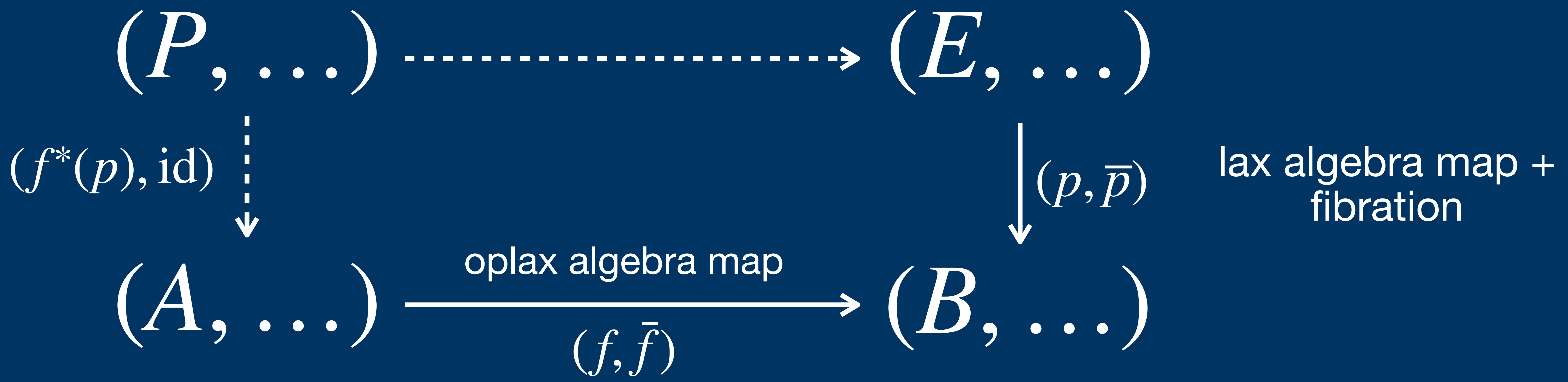

Then for the pullback $f^*(p) : P \to A$ in $\mathscr{C}$:

1. P acquires oplax algebra structure,
2. $f^*(p)$ is a strict algebra map, and
3. If $(p, \overline{p})$ is pseudo, this is a 2-pullback in $\mathrm{Oplax}\text{-}T\text{-}\mathrm{Alg}_{\mathrm{oplax}}$.

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…even when your models are diagrams

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If tight maps in $\mathscr{C}$ are closed under pullback (+ satisfy mild conditions), then in $[\mathscr{J}, \mathscr{C}]_{\mathrm{lax}}$ you can pull a tight map back along any map:

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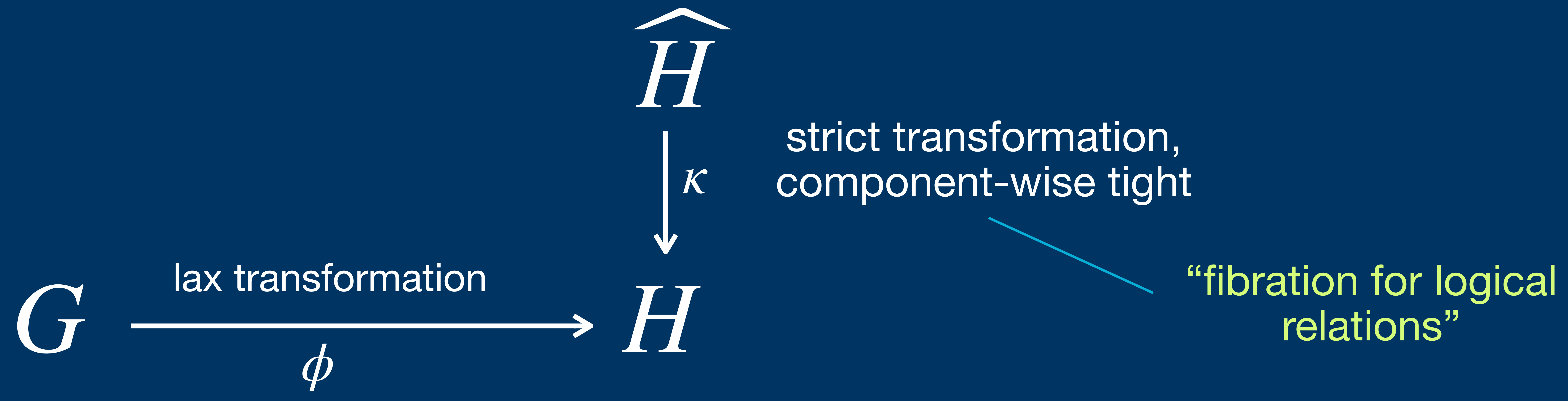

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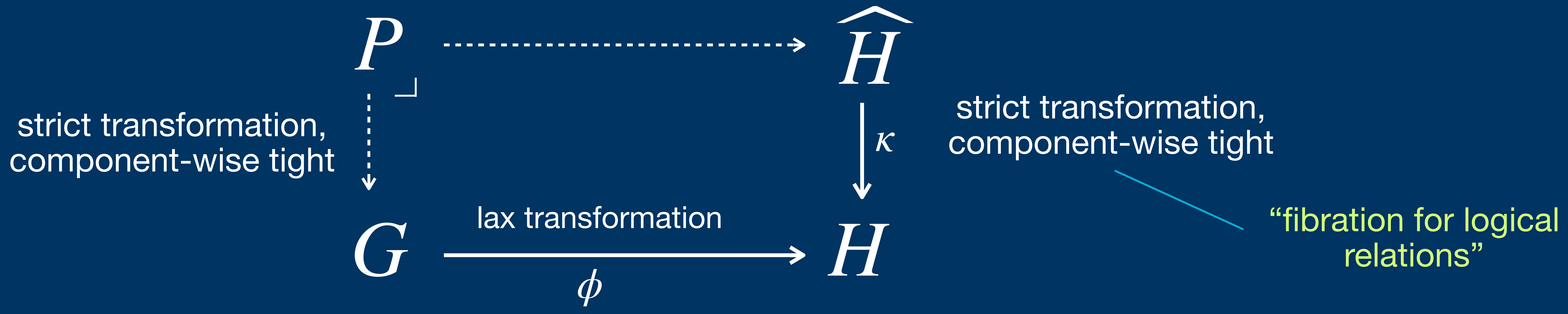

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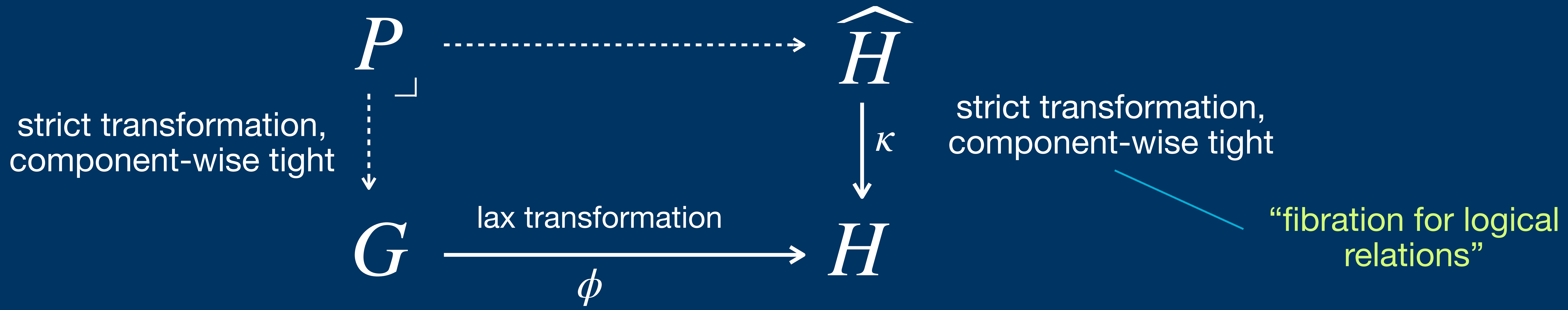

Moral: glueing works for monad models and adjunction models.

under slightly stronger conditions: can also take fibred products à la TT-lifting with multiple parameters

1. Fibrations internal to 2-categories tend to be the right thing*.
2. The theory of glueing works as you would hope, even in 2-categories of diagrams.

*sometimes sufficient but not necessary condition, but a lot of the theory still applies

Logical relations for CBPV models

(a recipe)

1. Define a 2-category LInd of locally-indexed categories,
2. Define CBPV as a sub-2-category of $[\mathrm{Adj}, \mathrm{LInd}]_{\mathrm{lax}}$,
3. Identify the fibrations internal to LInd,
4. Define a CBPV fibration to be a CBPV-morphism that's component-wise a LInd-fibration.

logical relations models + a glueing theorem follow from the algebra

a.k.a. Kripke relations of varying arity

Next steps: open logical relations

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characterise the definable maps in a model

[cf. Jung & Tiuryn]

semantic NbE à la *Semantic Analysis of Normalisation by Evaluation*

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Missing ingredient: a presheaf construction for CBPV models

2-categorical statement of Lafont argument

Open logical relations

(a.k.a. Kripke relations of varying arity)

Many properties of interest require us to pay attention to contexts.

For example: definability (Jung & Tiuryn)

$$\mathrm{Def}_\sigma(\Gamma) := \{ [\![M]\!] \mid \Gamma \vdash M : \sigma \} \subseteq \mathscr{M}([\![\Gamma]\!], [\![\sigma]\!])$$

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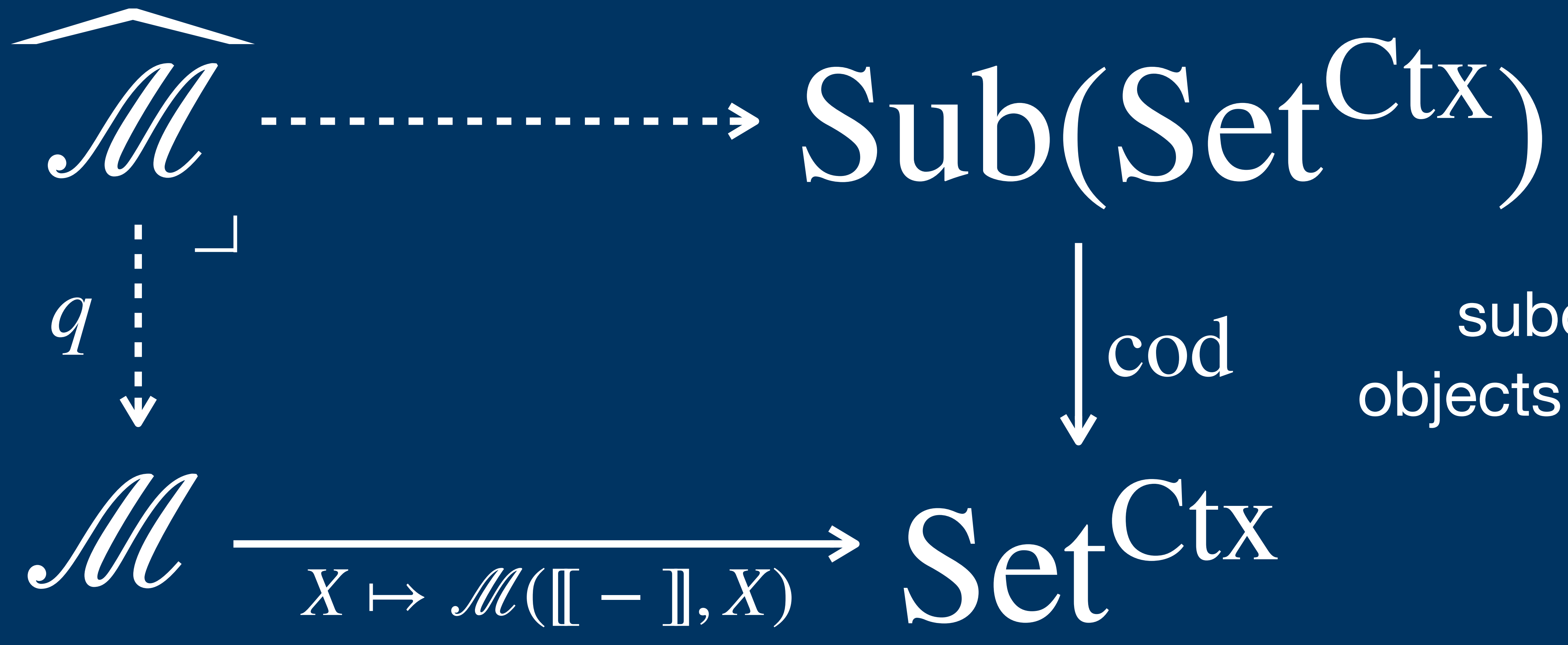
$$[\![\lambda x \,.\, M]\!] \in \mathrm{Def}_{\sigma \to \tau}(\underline{\Gamma}) \iff [\![M]\!] \in \mathrm{Def}_{\tau}(\underline{\Gamma, x : \sigma})$$

Open logical relations

(a.k.a. Kripke relations of varying arity)

objects:
$\big(X \in \mathscr{M}, R \hookrightarrow \mathscr{M}([\![-]\!], X)\big)$

i.e. families
$R(\Gamma) \subseteq \mathscr{M}([\![\Gamma]\!], X)$
indexed by contexts,
plus functoriality



semantic
model

subobject fibration:
objects are monos $\overline{P} \hookrightarrow P$

$\mathrm{Def}_\sigma(\Gamma) := \{[\![M]\!] \mid \Gamma \vdash M : \sigma\} \subseteq \mathscr{M}([\![\Gamma]\!], [\![\sigma]\!])$

open logical relations = glueing + presheaves

✔ ???

What about call-by-push-value?

= "locally indexed"

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First approach: use the enriched presheaf construction.
But: size issues become troubling.

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But: size issues become troubling.

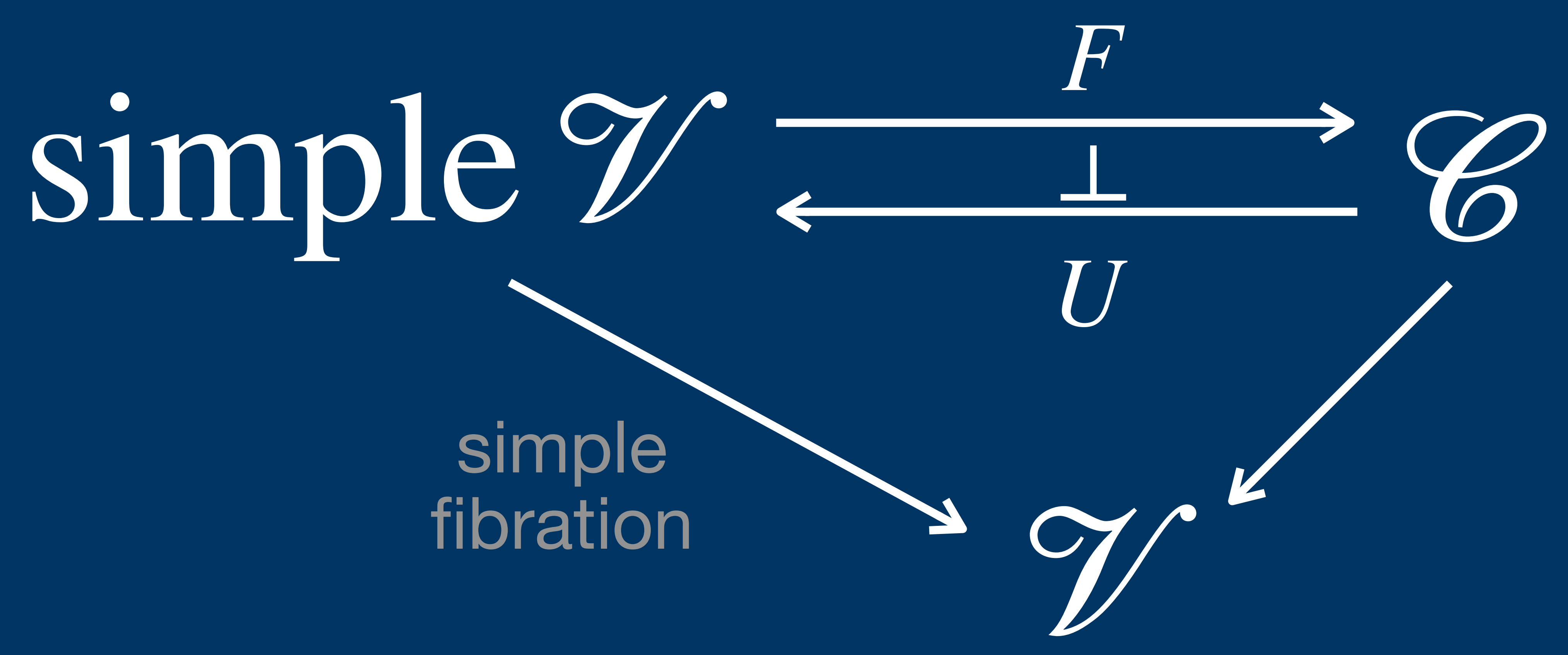
Alternative: recast CBPV models as certain fibrations.
(cf. Ahman, Vákár)

What about call-by-push-value?

each fibre has the same set of objects, and reindexing is ioo

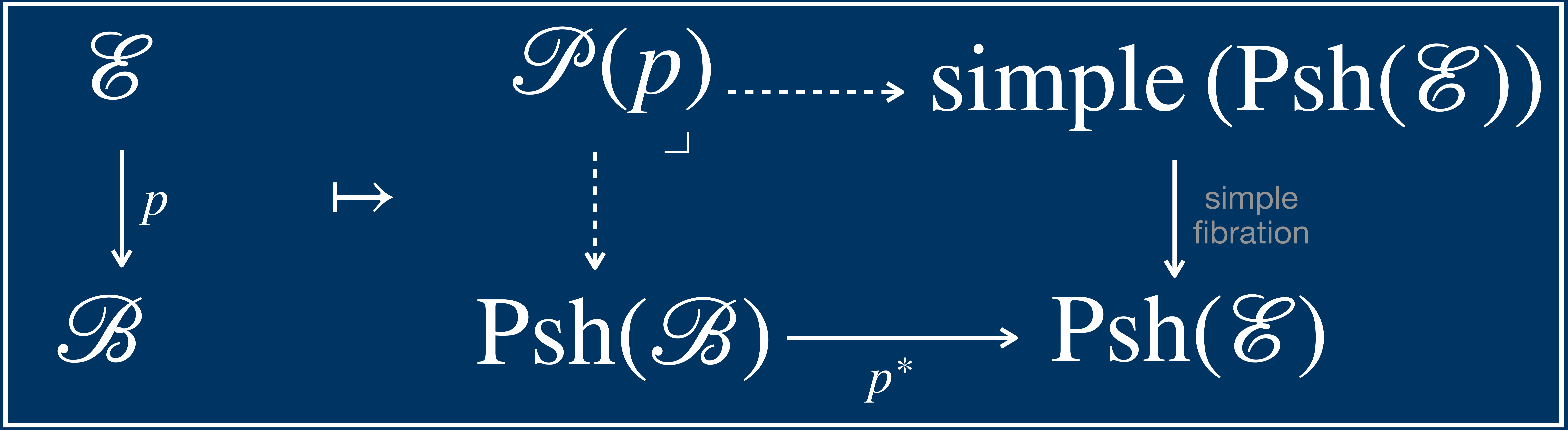
$\mathrm{Psh}(\mathscr{V})$-enriched category $\mathscr{C}$ $\overset{\text{folklore(?)}}{\longleftrightarrow}$ 'local' fibration $p : \mathscr{E} \to \mathscr{V}$

A CBPV-model is a (locally) fibred adjunction:



(cf. Ahman, Vákár)

The 2-category of local fibrations has a Yoneda structure



Induces a presheaf pseudofunctor $\rightsquigarrow$ sends adjunctions to adjunctions.

Determines a canonical presheaf operation on CBPV models.

Extracting the juice (WIP)

1. Open logical relations for CBPV: characterising definability, semantic NbE.
2. 2-categorical Lafont argument using this construction.
3. Seems to be quite a general way to build presheaf-like pseudomonads.

Summing up

1. Theory of logical relations includes fibrations for logical relations and glueing.
2. Fibrations internal to 2-categories of models seem to capture both aspects quite well.
3. Applies to examples where the models are more complex, e.g. CBPV.
4. Defining an appropriate presheaf construction should also lead to open logical relations, definability, NbE, conservativity, … .

the right abstraction (?)

follow what the algebra tells you

LICS paper link: arxiv.org/abs/2505.14482

Summing up

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Can we do the same for other languages?

Suggests a 'formal theory of logical relations' + applications.