A type theory for cartesian closed bicategories

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Cartesian closed bicategories

Cartesian closed categories ‘up to isomorphism’.

Examples:

- Generalised species and cartesian distributors
  particularly for applications in higher category theory
  (Fiore, Gambino, Hyland, Winskel), (Fiore & Joyal)
- Categorical algebra (operads)
  (Gambino & Joyal)
- Game semantics (concurrent games)
  (Yamada & Abramsky, Winskel *et al.*, Paquet)

Internal monoids

In a category with finite products:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

Unit law

$$\begin{array}{ccccc} 1 \times M & \xrightarrow{e \times M} & M \times M & \xleftarrow{M \times e} & M \times 1 \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle m} & \swarrow {\scriptstyle \cong} & \\ & & M & & \end{array}$$

Assoc. law

$$\begin{array}{ccccc} (M \times M) \times M & \xrightarrow{\cong} & M \times (M \times M) & \xrightarrow{M \times m} & M \times M \\ {\scriptstyle m \times M} \downarrow & & & & \downarrow {\scriptstyle m} \\ M \times M & & \xrightarrow[m]{\hspace{6em}} & & M \end{array}$$

In a category with finite products:

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In **Set**: monoids
In **Cat**: strict monoidal categories

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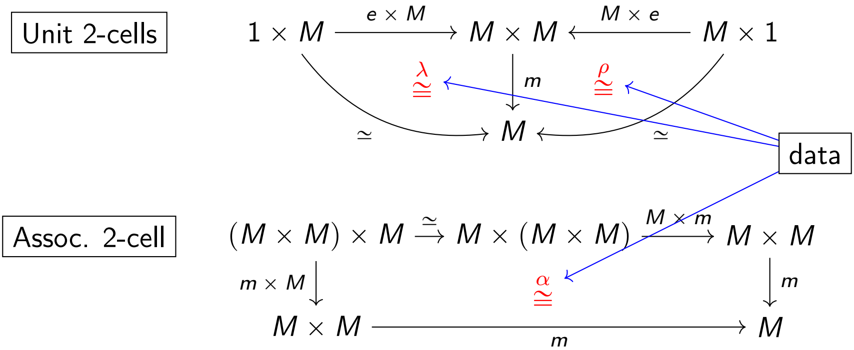

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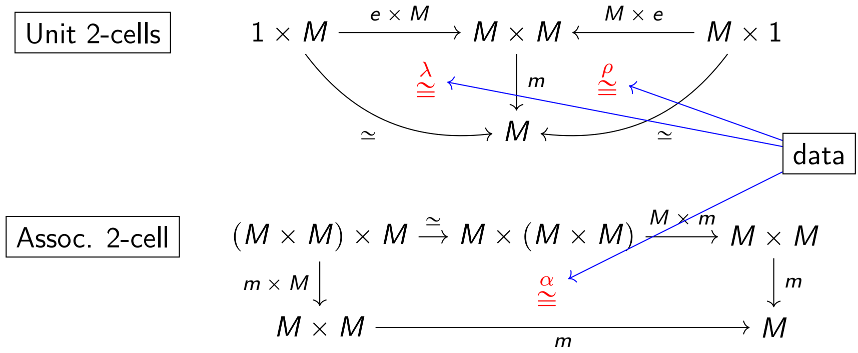


+ triangle and pentagon laws $\rightsquigarrow$ monoidal category

Internal pseudomonoids

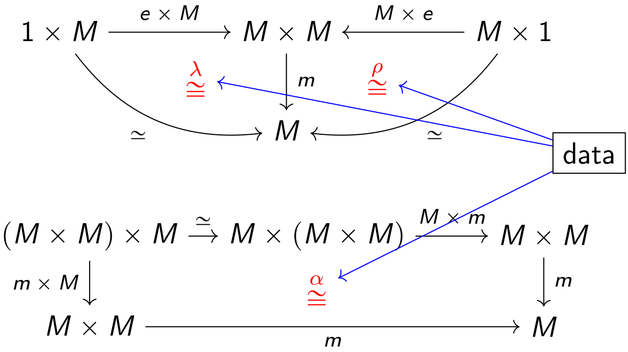
In **Cat**:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

...likewise in any fp-bicategory

Unit 2-cells

Assoc. 2-cell



+ triangle and pentagon laws ⇝ monoidal category

In a CCC every $[X \Rightarrow X]$ becomes a monoid:

$$\left(1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\right)$$

? In a cc-bicategory every $[X \Rightarrow X]$ becomes a pseudomonoid:

$$\left(1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\right)$$

need to check
coherence laws
(*i.e.* triangle + pentagon)

Coherence

Programme:

1. Construct a type theory $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$ for cartesian closed bicategories
   (this work),
2. Use NBE to prove the type theory is coherent
   bicategorical version of [Fiore2002]
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Application:

Algebraic structure definable in every CCC

$\Rightarrow$ algebraic pseudo-structure definable in every cc-bicategory

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1. Generalises the simply-typed lambda calculus,
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3. Is *sound* and *complete*
   i.e. freeness property for the syntactic model.

Bicategories

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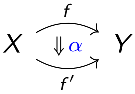
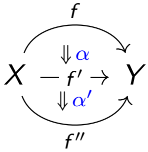
  2-cells 



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- Functors

$$\mathbf{1} \xrightarrow{\mathrm{Id}_X} \mathcal{B}(X, X)$$

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$$X \underset{f'}{\overset{f}{\Downarrow \alpha}} Y \underset{g'}{\overset{g}{\Downarrow \beta}} Z$$

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- Invertible 2-cells

$$(h \circ g) \circ f \xRightarrow{\mathbf{a}_{h,g,f}} h \circ (g \circ f)$$

$$\mathrm{Id}_X \circ f \xRightarrow{\mathbf{l}_f} f$$

$$g \circ \mathrm{Id}_X \xRightarrow{\mathbf{r}_g} g$$

subject to a triangle law and pentagon law.

Bicategories $\mathcal{B}$ equipped with *biuniversal* 1-cells

Cartesian closed bicategories

Bicategories $\mathcal{B}$ equipped with *biuniversal* 1-cells

$$\text{(fp)} \qquad \pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$$

$$\text{(cc)} \qquad \mathrm{eval} : (A \Rightarrow B) \times A \to B$$

NB: Differ from the 'cartesian bicategories' of Carboni and Walters!

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inducing families of equivalences

$$\mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \quad \simeq \quad \prod_{i=1}^{n} \mathcal{B}(X, A_i)$$

$$\mathcal{B}(X, A \Rightarrow B) \quad \simeq \quad \mathcal{B}(X \times A, B)$$

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$$\mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \underset{\langle -,\ldots,= \rangle}{\overset{(\pi_1 \circ -,\ldots,\pi_n \circ -)}{\rightleftarrows}} \prod_{i=1}^{n} \mathcal{B}(X, A_i) \quad \perp\simeq$$

(tupling)

$$\mathcal{B}(X, A \Rightarrow B) \underset{\lambda}{\overset{\mathrm{eval}_{A,B} \circ (- \times A)}{\rightleftarrows}} \mathcal{B}(X \times A, B) \quad \perp\simeq$$

(currying)

NB: Differ from the 'cartesian bicategories' of Carboni and Walters!

Substitution and composition

In any CCC:

$$[\![x_k[u_1/x_1, \dots, u_n/x_n]]\!] = [\![u_k]\!] = \pi_k \circ \langle [\![u_1]\!], \dots, [\![u_n]\!] \rangle$$

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Question: what is bicategorical substitution?

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such that

$$\begin{aligned} x_k[u_i/x_i] &= u_k \qquad\qquad (1 \leqslant k \leqslant n) \\ t[x_i/x_i] &= t \\ t[u_i/x_i][v_j/y_j] &= t\,[u_i[v_j/y_j]/x_i] \end{aligned}$$

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- Structural isomorphisms

$$\mathsf{p}^{(k)}[u_1, \ldots, u_n] \xRightarrow{\varrho^{(k)}_{u_\bullet}} u_k \qquad (1 \leqslant k \leqslant n)$$

$$t[\mathsf{p}^{(1)}, \ldots, \mathsf{p}^{(n)}] \xRightarrow{\iota_t} t$$

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subject to a triangle law and pentagon law.

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Note: every biclone defines a bicategory

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Vertical composition:
$$\frac{\Gamma \vdash \tau' : t' \Rightarrow t'' : B \qquad \Gamma \vdash \tau : t \Rightarrow t' : B}{\Gamma \vdash \tau' \bullet \tau : t \Rightarrow t'' : B}$$

Identities:
$$\frac{\Gamma \vdash t : B}{\Gamma \vdash \mathrm{id}_t : t \Rightarrow t : B}$$

A type theory for biclones

A *substitution* functor

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \prod_{i=1}^{n} \mathbb{C}(\Gamma; X_i) \to \mathbb{C}(\Gamma; Y)$$
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Explicit substitution:

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash t : B \qquad (\Delta \vdash u_i : A_i)_{i=1.,n}}{\Delta \vdash t\,\{x_i \mapsto u_i\} : B}$$

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash \tau : t \Rightarrow t' : B \qquad (\Delta \vdash \sigma_i : u_i \Rightarrow u'_i : A_i)_{i=1,\ldots,n}}{\Delta \vdash \tau\,\{x_i \mapsto \sigma_i\} : t\,\{x_i \mapsto u_i\} \Rightarrow t'\,\{x_i \mapsto u'_i\} : B}$$

$\rightsquigarrow$ binds the variables $x_1, \ldots, x_n$

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Structural isomorphisms $\varrho^{(k)}, \iota, \mathsf{assoc}$

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Distinguished invertible rewrites *e.g.*:

$$\frac{(\Delta \vdash u_i : A_i)_{i=1,\ldots,n}}{x_1 : A_1, \ldots, x_n : A_n \vdash \varrho^{(k)}_{u_\bullet} : x_k \{x_i \mapsto u_i\} \overset{\cong}{\Longrightarrow} u_k : A_k} \ (1 \leqslant k \leqslant n)$$

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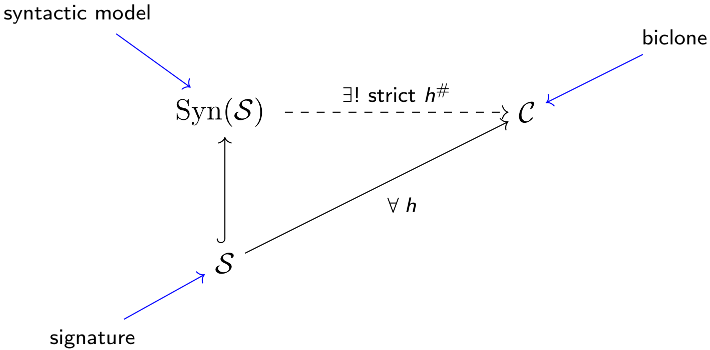
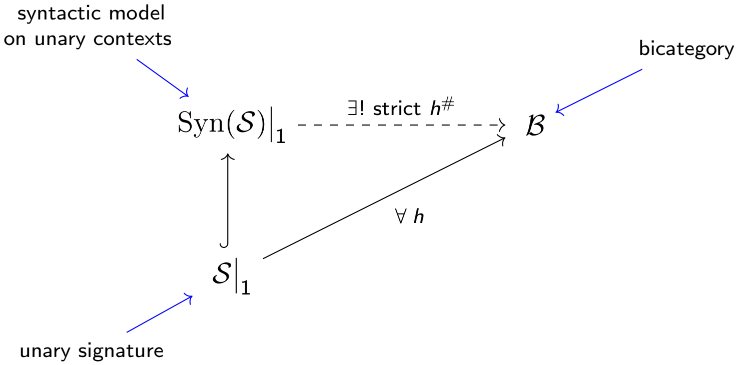


The syntactic model is free



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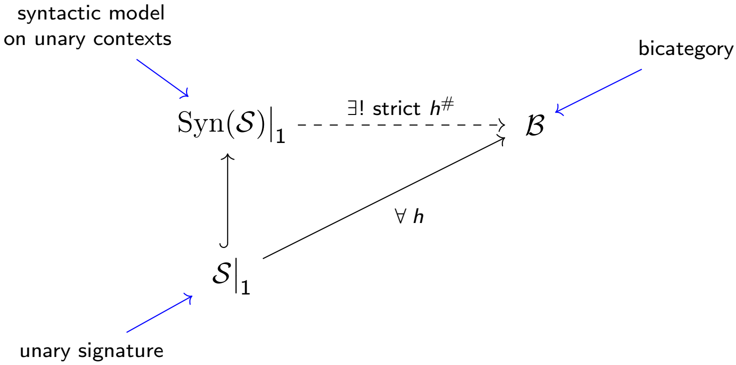

$\rightsquigarrow$ An internal language for bicategories.

fp-Bicategories

fp-Bicategories

1-cells

$$\pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$$

Adjoint equivalences

$$\mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \underset{\langle -,\ldots,= \rangle}{\overset{(\pi_1 \circ -,\ldots,\pi_n \circ -)}{\rightleftarrows}} \perp\simeq \quad \prod_{i=1}^{n} \mathcal{B}(X, A_i)$$

A type theory for fp-bicategories

1-cells $\pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$

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Projections $\dfrac{}{p : \Pi_n(A_1, \ldots, A_n) \vdash \pi_i(p) : A_i}\ (1 \leqslant i \leqslant n)$

A type theory for fp-bicategories

$$\text{Equivalences} \quad \mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \underset{\langle -, \ldots, = \rangle}{\overset{(\pi_1 \circ -, \ldots, \pi_n \circ -)}{\rightleftarrows}} \perp\simeq \prod_{i=1}^{n} \mathcal{B}(X, A_i)$$

A type theory for fp-bicategories

Equivalences $\mathcal{B}(X, \Pi_n(A_1, \dots, A_n)) \underset{\langle -,\dots,= \rangle}{\overset{(\pi_1 \circ -,\dots,\pi_n \circ -)}{\perp\simeq}} \prod_{i=1}^{n} \mathcal{B}(X, A_i)$

$$\varpi^{(i)} \bullet (\pi_i \circ (-)) \left(\quad \dfrac{\pi_i \circ u \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \langle t_1, \dots, t_n \rangle : \Pi_n(A_1, \dots, A_n)} \quad \right) \mathsf{p}^{\dagger}(-, \dots, =)$$

for a counit $(\varpi^{(i)} : \pi_i \circ \langle t_1, \dots, t_n \rangle \Rightarrow t_i : A_i)_{i=1,\dots,n}$

A type theory for fp-bicategories

Equivalences $\mathcal{B}\left(X, \Pi_n(A_1, \dots, A_n)\right) \underset{\langle -,\dots,= \rangle}{\overset{(\pi_1 \circ -, \dots, \pi_n \circ -)}{\rightleftarrows}} \perp\simeq \prod_{i=1}^{n} \mathcal{B}(X, A_i)$

syntactic sugar

$$\varpi^{(i)} \bullet \pi_i\{(-)\} \Bigg(\frac{\pi_i\{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \mathrm{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)} \Bigg) \, \mathrm{p}^{\dagger}(-, \dots, =)$$

for a counit $\left(\varpi^{(i)} : \pi_i\{\mathrm{tup}(t_1, \dots, t_n)\} \Rightarrow t_i : \Pi_n(A_1, \dots, A_n)\right)_{i=1,\dots,n}$

A type theory for fp-bicategories

$$\varpi^{(i)} \bullet \pi_i \{(-)\} \left(\quad \frac{\pi_i \{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \mathsf{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)} \quad \right) \mathsf{p}^{\dagger}(-, \dots, =)$$

A type theory for fp-bicategories

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Tupling map $$\frac{(\Gamma \vdash t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \mathsf{tup}(t_1 \dots, t_n) : \Pi_n(A_1, \dots, A_n)}$$

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Counit (β-law)
$$\frac{(\Gamma \vdash t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \varpi^{(k)}_{t_\bullet} : \pi_k\{\mathrm{tup}(t_1 \dots, t_n)\} \overset{\cong}{\Longrightarrow} t_k : A_k} \; (1 \leqslant k \leqslant n)$$

A type theory for fp-bicategories

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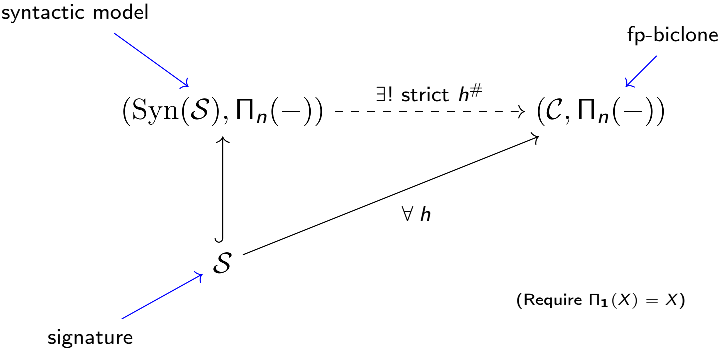
Mediating 2-cell
$$\frac{(\Gamma \vdash \alpha_i : \pi_i\{u\} \Rightarrow t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \mathrm{p}^{\dagger}(\alpha_1, \dots, \alpha_n) : u \Rightarrow \mathrm{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)}$$

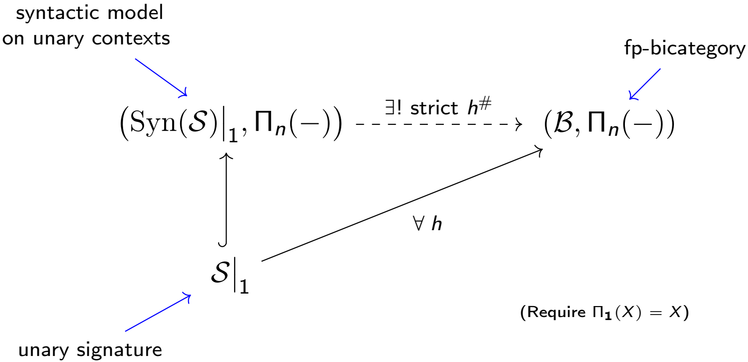
+ three equational rules. $\rightsquigarrow$ η-law is derivable

The syntactic model is free



(Require $\Pi_{\mathbf{1}}(X) = X$)

The syntactic model is free



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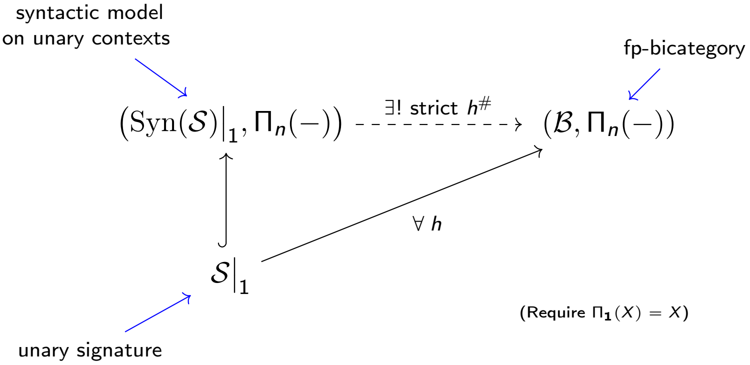
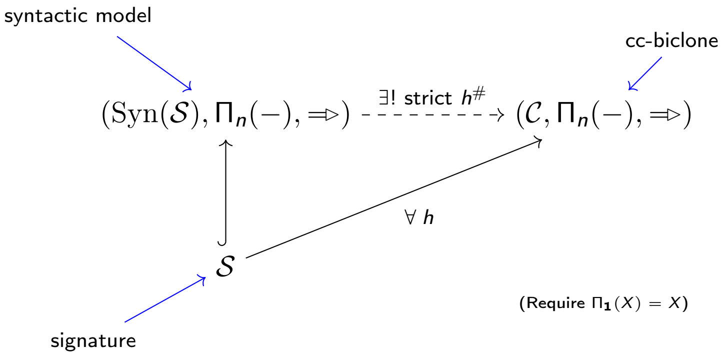


$\rightsquigarrow$ An internal language for fp-bicategories.

derived from definition of biadjoint

The syntactic model is free



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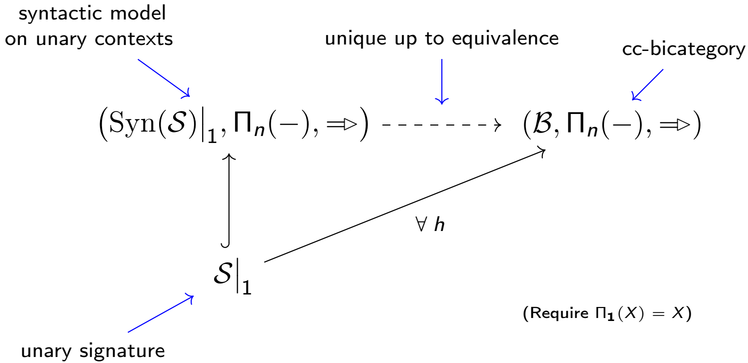
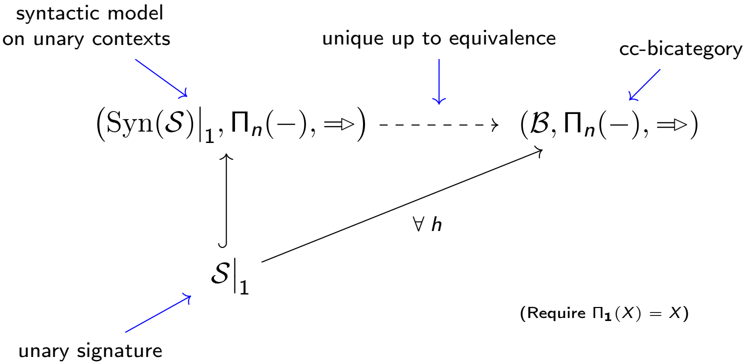

syntactic model
on unary contexts
unique up to equivalence
cc-bicategory
$(\mathrm{Syn}(\mathcal{S})\big|_1, \Pi_n(-), \Rightarrow)$
$(\mathcal{B}, \Pi_n(-), \Rightarrow)$
$\forall\, h$
$\mathcal{S}\big|_1$
unary signature
(Require $\Pi_{\mathbf{1}}(X) = X$)

The syntactic model is free



(Require $\Pi_{\mathbf{1}}(X) = X$)

$\rightsquigarrow$ An internal language for cartesian closed bicategories.

STLC up to isomorphism

Embedding of STLC-terms to $\Lambda_{\mathrm{ps}}^{\times,\to}$-terms:

$$\begin{aligned} x_k &\mapsto x_k \\ \pi_k(t) &\mapsto \pi_k\,\{\llparenthesis\, t \,\rrparenthesis\} \\ \langle t_1, \ldots, t_n \rangle &\mapsto \mathsf{tup}(\llparenthesis\, t_1 \,\rrparenthesis, \ldots, \llparenthesis\, t_n \,\rrparenthesis) \\ \mathsf{app}(t, u) &\mapsto \mathsf{eval}\,\{\llparenthesis\, t \,\rrparenthesis, \llparenthesis\, u \,\rrparenthesis\} \\ \lambda x.t &\mapsto \lambda x.\llparenthesis\, t \,\rrparenthesis \end{aligned}$$

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$$\left(\text{STLC terms } \Gamma \vdash t : B\right)\big/_{\beta\eta} \cong \left(\Lambda_{\mathrm{ps}}^{\times,\to}\text{-terms } \Gamma \vdash t : B\right)\big/_{\cong_B^\Gamma}$$

$t \cong_B^\Gamma t' \;\Leftrightarrow\; \Gamma \vdash \tau : t \overset{\cong}{\Longrightarrow} t' : B$
for some invertible τ

Key properties of $\Lambda_{\text{ps}}^{\times,\rightarrow}$:

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1. Principled development $\Rightarrow$ few rules,
2. An internal language for cc-bicategories,
3. STLC up-to-isomorphism.

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1. Principled development $\Rightarrow$ few rules,
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3. STLC up-to-isomorphism.

A type theory for cartesian closed bicategories (LICS'19):
`https://arxiv.org/abs/1904.06538`

1-cells

$$\mathrm{eval}_{A,B} : (A \Rightarrow B) \times A \to B$$

Adjoint equivalences

$$\mathcal{B}(X, A \Rightarrow B) \quad \underset{\lambda}{\overset{\mathrm{eval}_{A,B} \circ (- \times A)}{\rightleftarrows}} \perp\simeq \quad \mathcal{B}(X \times A, B)$$

Rules for exponentials

explicit weakening by x → $(x : A)$ ← free variable in context

$$\epsilon \bullet \mathrm{eval}\{(-)\{\mathrm{inc}_x\}, x\} \quad \frac{\mathrm{eval}\{u\{\mathrm{inc}_x\}, x\} \Rightarrow t : B}{u \Rightarrow \lambda x.t : A \Rightarrow B} \quad \mathrm{e}^\dagger(x \,.\, -)$$

$$\frac{}{f : A \Rightarrow B, x : A \vdash \mathrm{eval}(f, x) : B}\ \mathrm{eval} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\ \mathrm{lam}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma, x : A \vdash \epsilon_t : \mathrm{eval}\{(\lambda x.t)\{\mathrm{inc}_x\}, x\} \Rightarrow t : B}\ \epsilon\text{-intro } (\beta\text{-rule})$$

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A \Rightarrow B \qquad \Gamma, x : A \vdash \alpha : \mathrm{eval}\{u\{\mathrm{inc}_x\}, x\} \Rightarrow t : B}{\Gamma \vdash \mathrm{e}^\dagger(x \,.\, \alpha) : u \Rightarrow \lambda x.t : A \Rightarrow B}\ \mathrm{e}^\dagger(x \,.\, \alpha)\text{-intro}$$

\+ three equational rules $\rightsquigarrow$ η-rule derivable