Constructing fully-abstract models of effectful λ-calculi

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preprint at cs.ox.ac.uk/people/philip.saville/home.html

Slogan: models* go in, fully abstract models come out

Contextual equivalence [Morris, Milner,. . .]

$$\Gamma \vdash M \simeq_{\mathrm{ctx}} M' : \sigma \qquad \Longleftrightarrow \qquad \mathcal{C}[M] \Downarrow V \iff \mathcal{C}[M'] \Downarrow V$$

$\mathcal{C}[-]$ any closed ground context

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doesn't affect
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Reasoning about $\simeq_{\mathrm{ctx}}$ is hard! Want semantic techniques round this

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How does semantic equality relate to $\simeq_{\mathrm{ctx}}$?

Adequacy: $[\![M]\!] = [\![M']\!] \implies M \simeq_{\mathrm{ctx}} M'$

Full abstraction: $M \simeq_{\mathrm{ctx}} M' \implies [\![M]\!] = [\![M']\!]$

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In an adequate, fully abstract model
semantic equality characterises contextual equivalence

The quest for full abstraction

late 1960s	roots of the definition
1975	Milner introduces full abstraction
1977	Plotkin: domains model for PCF is not fully abstract
80s & 90s	attempts to classify "sequentiality" + lots more
late 90s	games models,
	O'Hearn & Riecke's domains + logical relations model
21st C	Marz, Riecke, Ehrhard *et al.*, Matache *et al.*, ...
	$\rightsquigarrow$ mainly focussed on languages with recursion

This work: cranking the handle

signature S

= chosen base types, effect operations, & primitives

+

semantic model $(\mathcal{M}, T, s)$

= CCC with coproducts $\mathcal{M}$
+ strong monad T
+ interpretation s
+ conditions on $\mathcal{M}$, s

fully abstract model $\mathrm{OHR}(\mathcal{M})$
of computational λ-calculus + constants + sums

inspired by O'Hearn & Riecke's model

concrete over $\mathcal{M}$:
maps in $\mathrm{OHR}(\mathcal{M})$ are
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determines an interpretation
$s[\![\Gamma \vdash M : \sigma]\!] : s[\![\Gamma]\!] \to Ts[\![\sigma]\!]$

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concrete over $\mathcal{M}$:
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e.g. small subcategory Set_κ of Set
+ any monad
+ any interpretation
with a constant $\underline{b}$ for every $b \in s[\![\beta]\!]$

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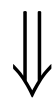

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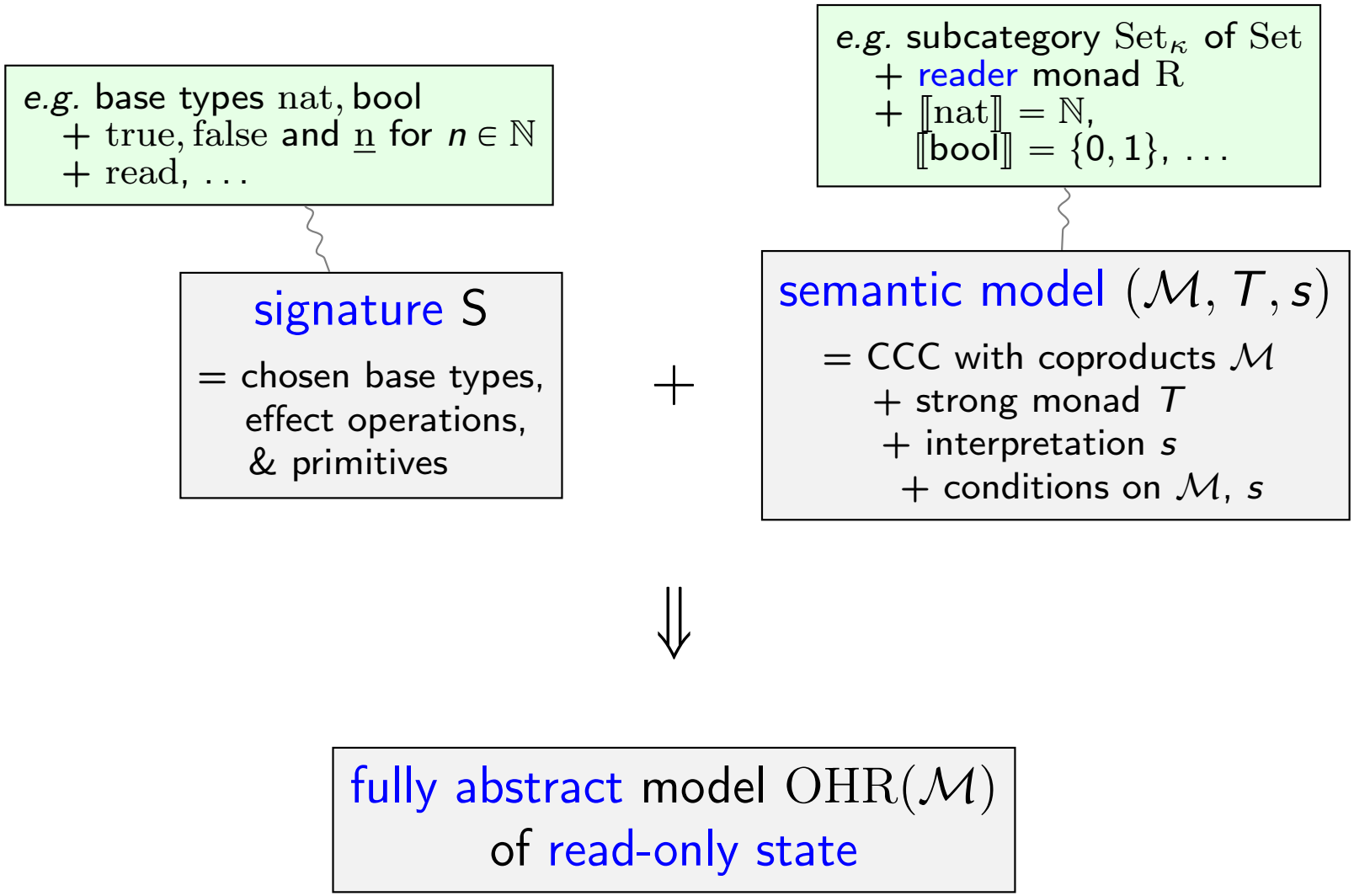
e.g. base types nat, bool
+ $\mathrm{true}, \mathrm{false}$ and $\underline{\mathrm{n}}$ for $n \in \mathbb{N}$
+ read, ...
e.g. subcategory Set_κ of Set
+ reader monad R
+ $[\![\mathrm{nat}]\!] = \mathbb{N}$,
$[\![\mathrm{bool}]\!] = \{0, 1\}$, ...
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= chosen base types,
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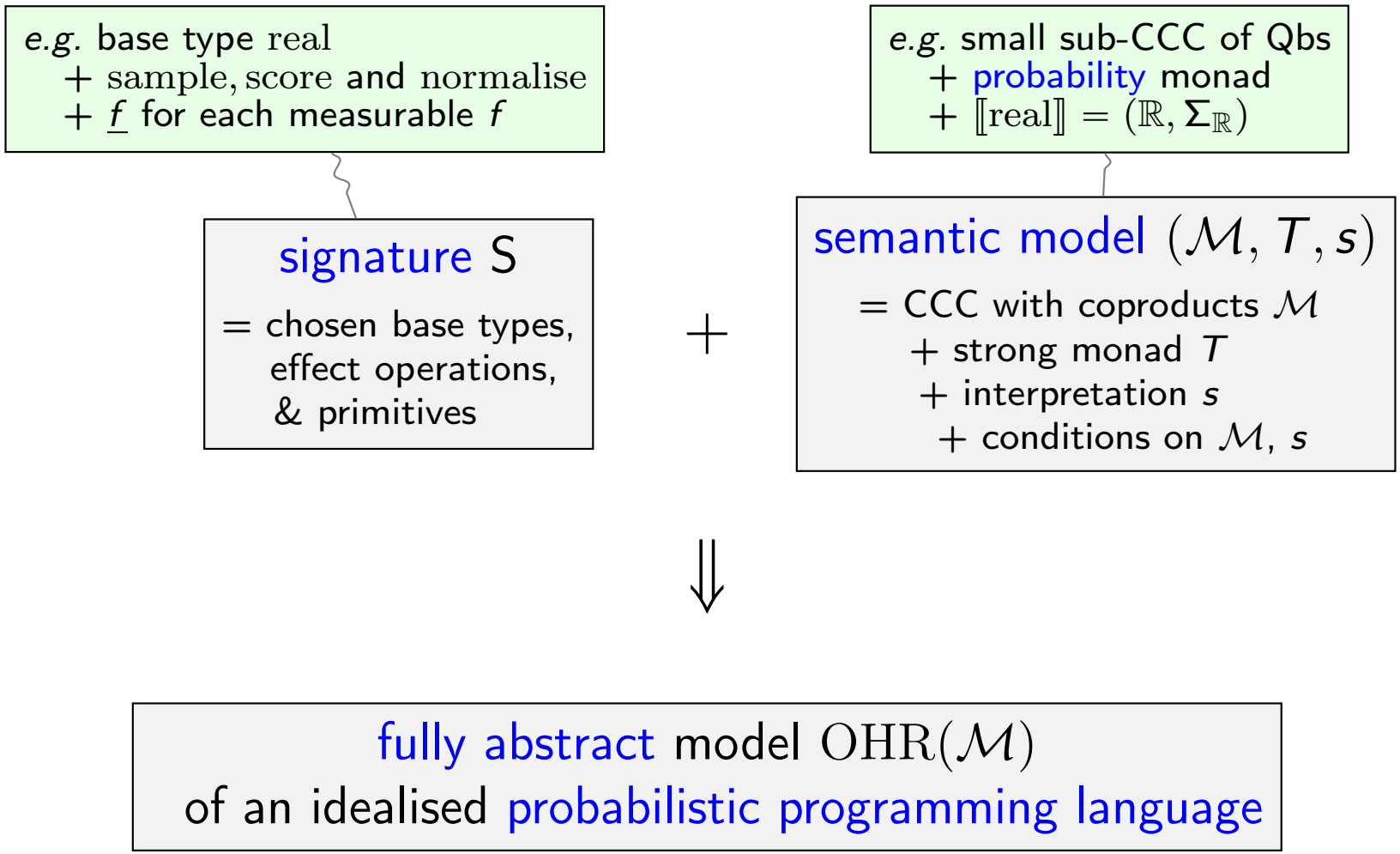
fully abstract model $\mathrm{OHR}(\mathcal{M})$
of read-only state


e.g. base type real
+ sample, score and normalise
+ f for each measurable f
e.g. small sub-CCC of Qbs
+ probability monad
+ [[real]] = (ℝ, Σ_ℝ)
signature S
= chosen base types,
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semantic model (M, T, s)
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⇓
fully abstract model OHR(M)
of an idealised probabilistic programming language

The big picture

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$\exists$ morphisms in $\mathcal{M}$ expressing behaviour the syntax cannot

[*c.f.* parallel-or]

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Solution:

remove all such counterexamples to contextual equivalence

Up next: a recipe that doesn't quite work

. . . but is the template for our construction

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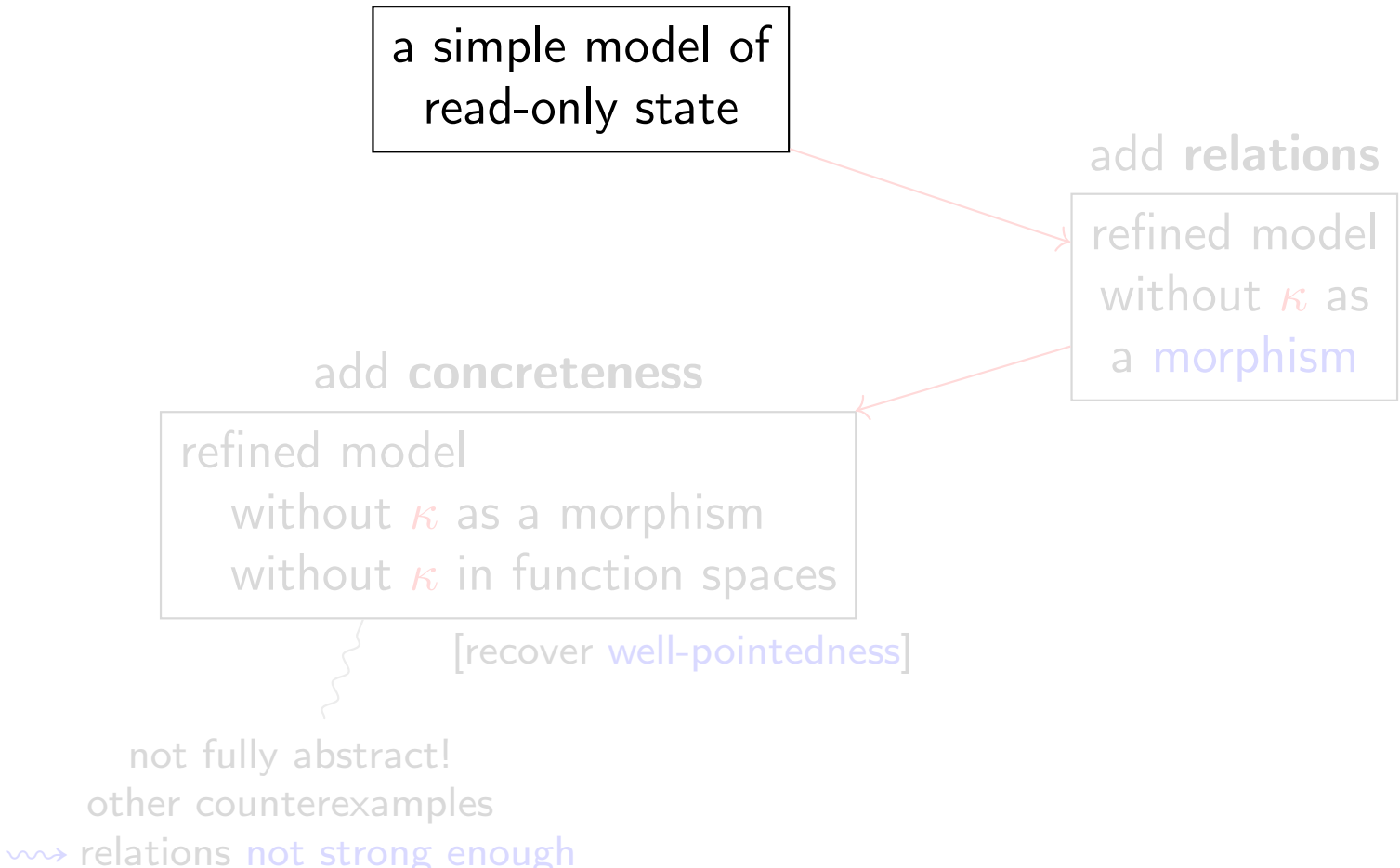

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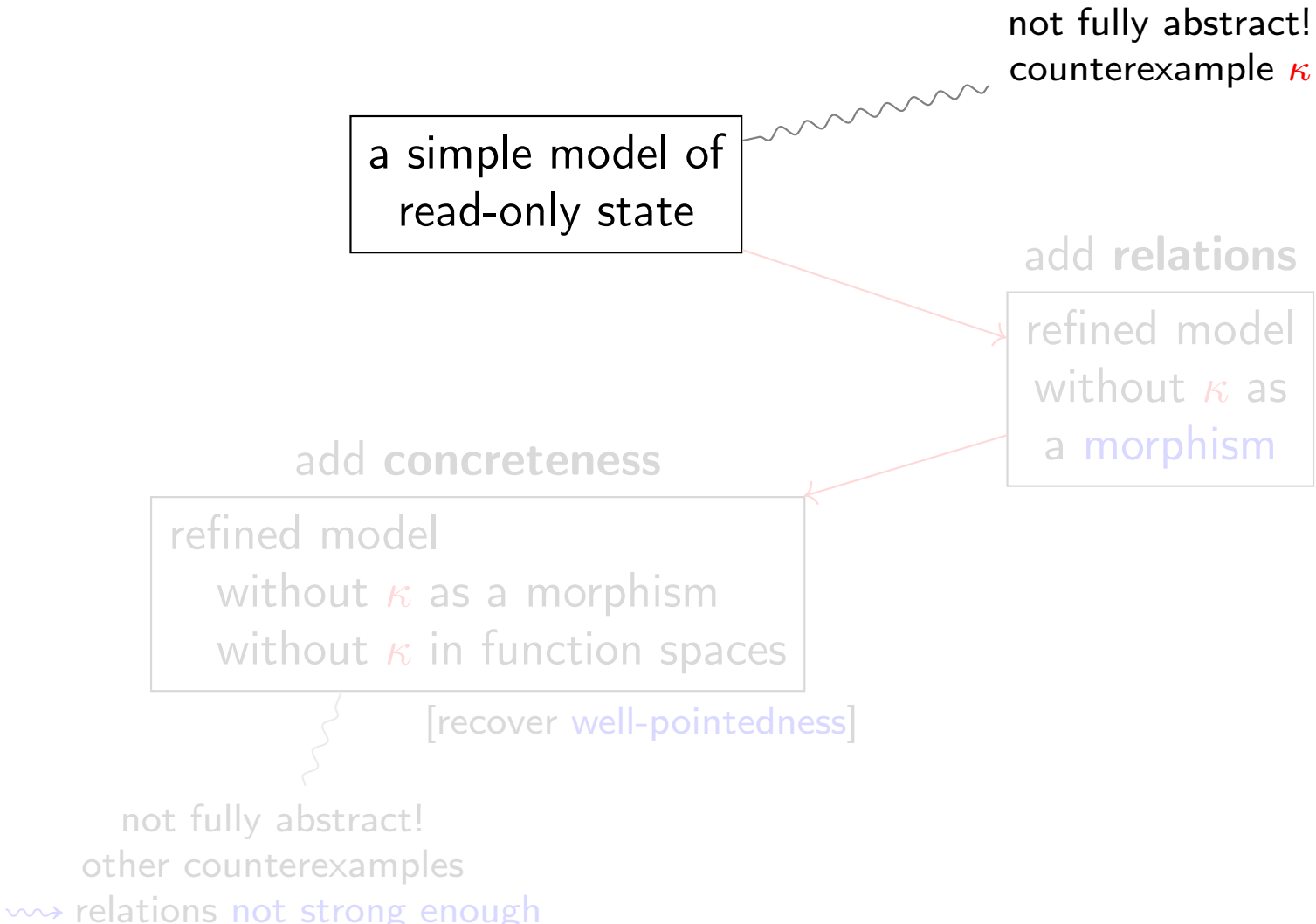

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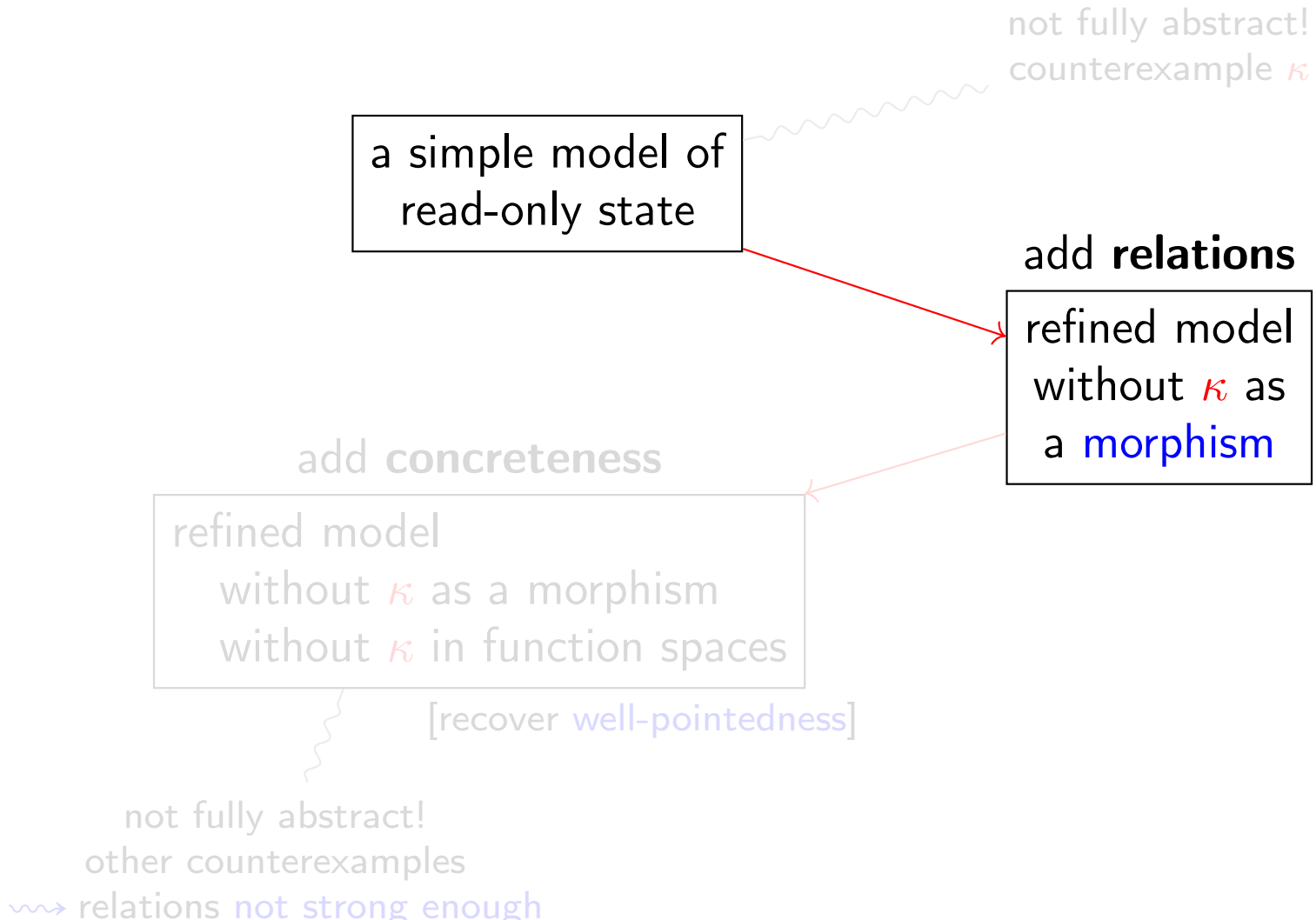

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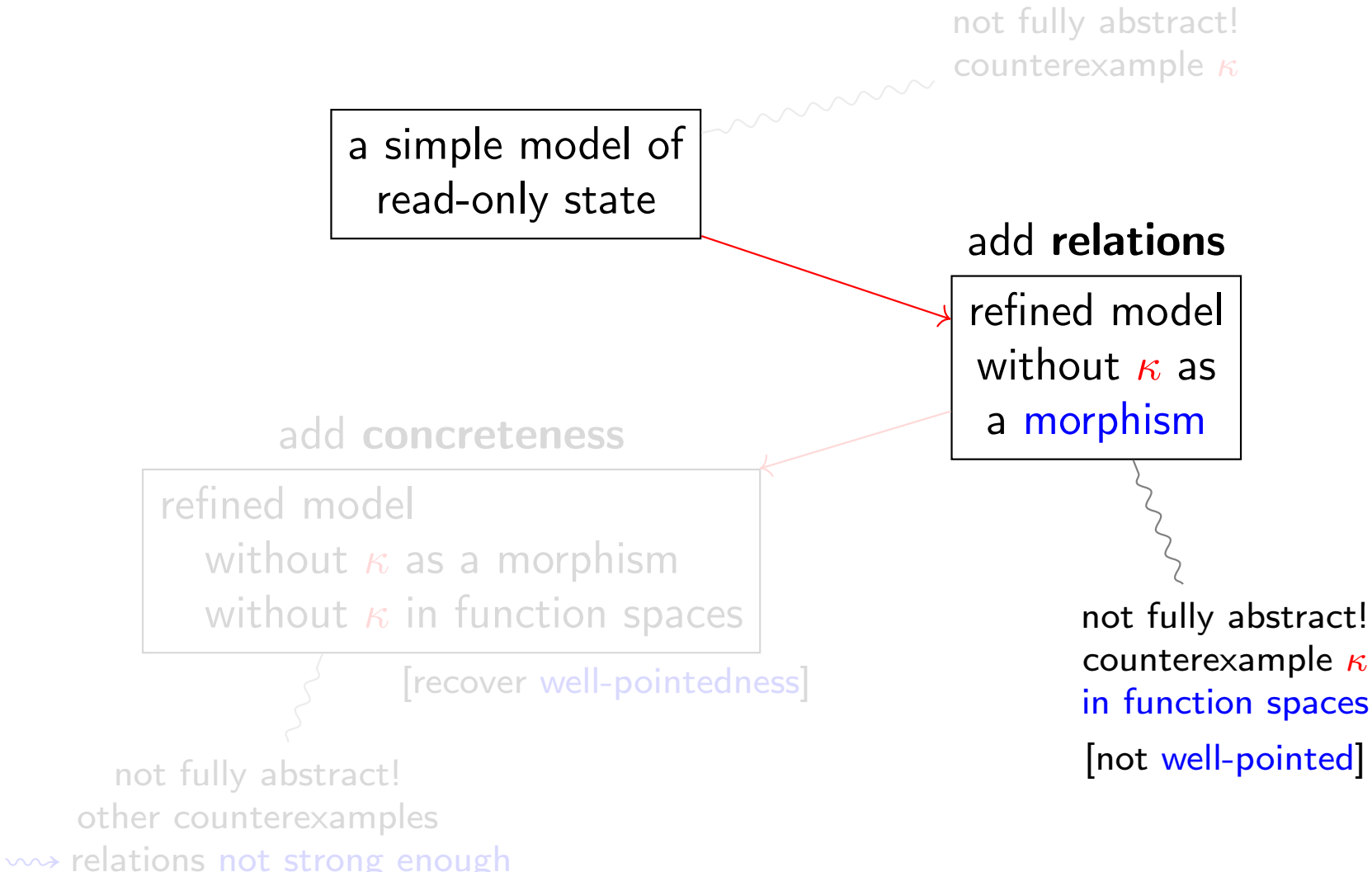

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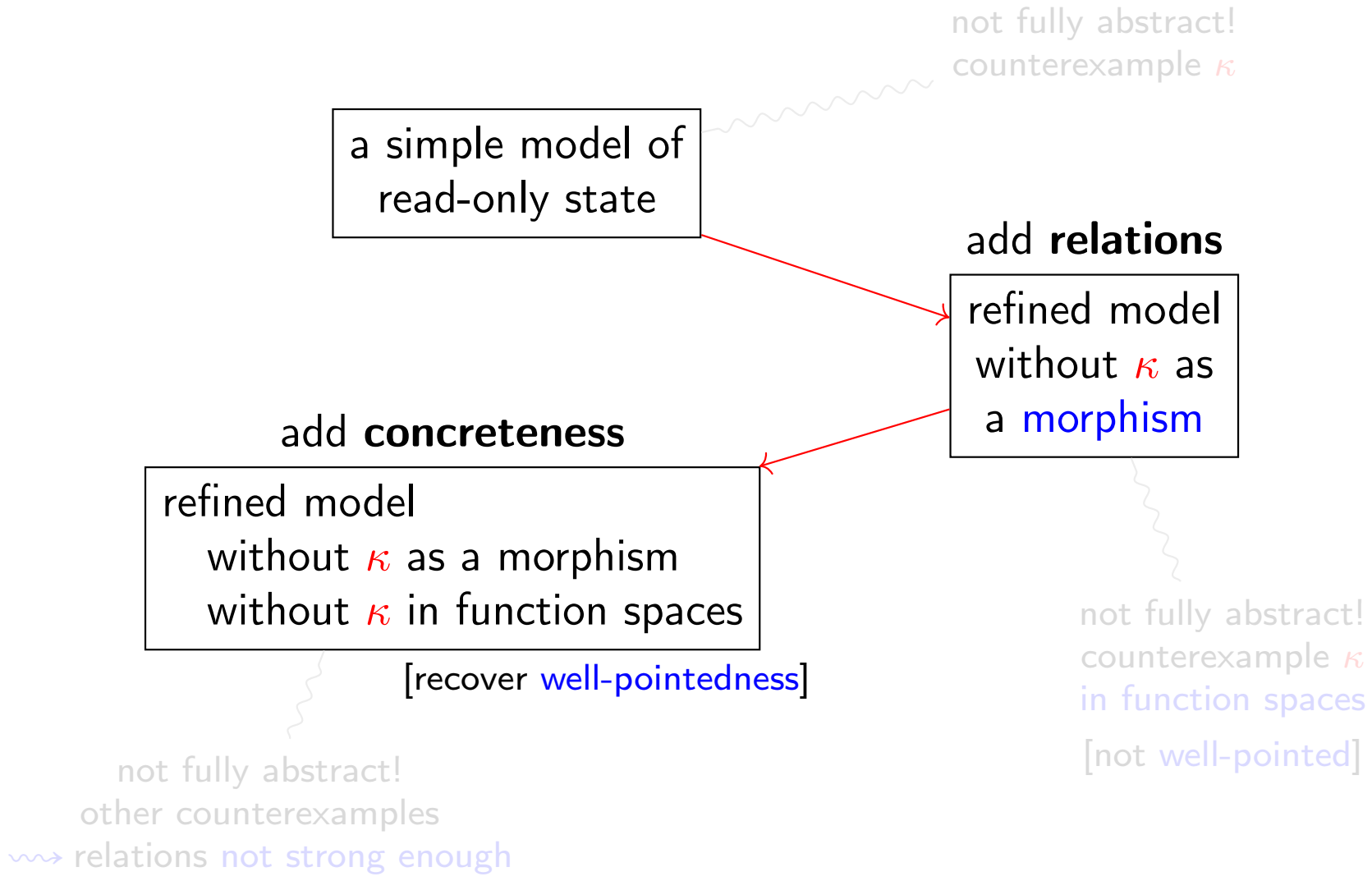

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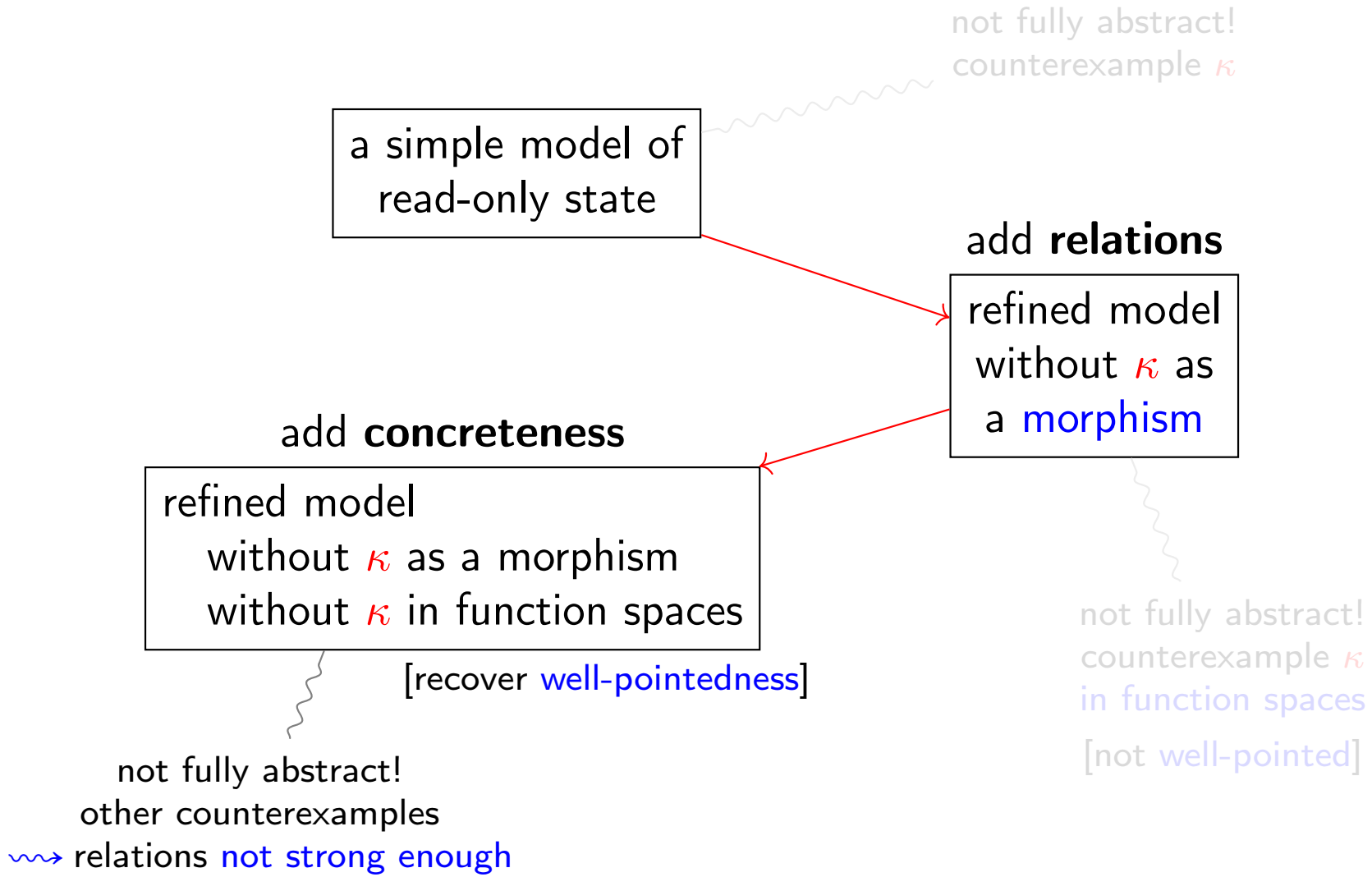

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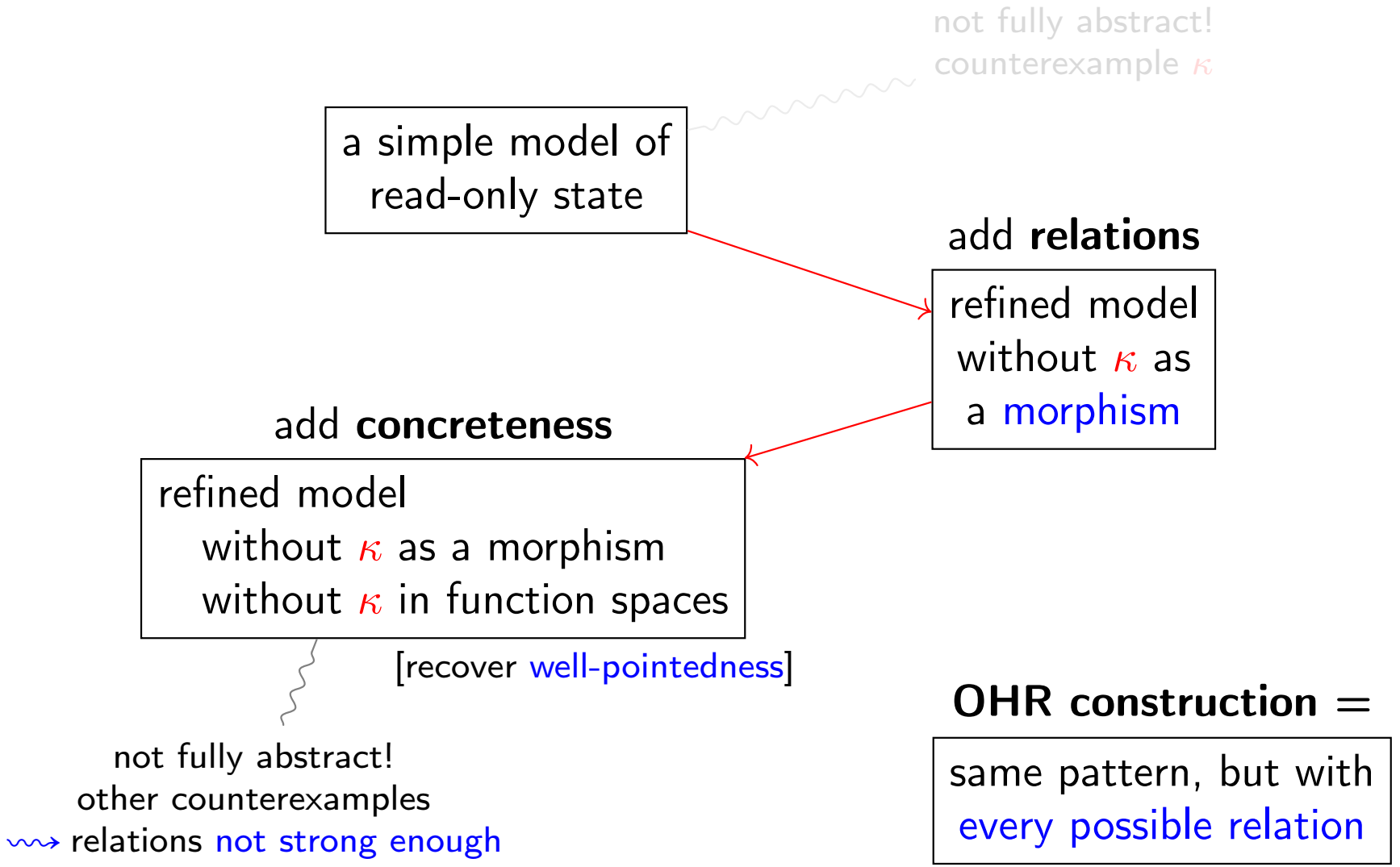

A motivating example: read-only state [Matache & Staton]

Idea [omitting sums for now]

1. A global, one-bit memory cell
2. You can read, but not write

The signature

types $\tau ::= \texttt{bool} \mid \tau * \tau \mid 1 \mid \tau \to \tau$

types τ ::= bool | τ * τ | 1 | $\tau \to \tau$

terms M ::=

STLC

x	# variables
| (M,M)	# product types
| π_i(M)	
| ()	
| λx . M	# function types
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    | M M
# primitives
    | tt                         # boolean values
    | ff
    | ∧                          # boolean operations
    | ∨
    | ¬
    | if M then M else M         # branching
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   | if M then M else M          # branching
# effect operations
   | read : 1 → bool             # read from the cell
```

The semantic model

A model $(\mathrm{Fin}, \mathrm{R}, \boldsymbol{s})$

Idea:

1. Interpret programs as functions;
2. Parametrise by what's in the cell.

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- Use the natural interpretation:

$$s[\![\mathsf{bool}]\!] := 2 := \{0, 1\}$$

$$s[\![\diamond \vdash \mathtt{tt} : \mathsf{bool}]\!] = \mathrm{const}_1$$

$$s[\![\diamond \vdash \mathtt{ff} : \mathsf{bool}]\!] = \mathrm{const}_0$$

$$s[\![\Gamma \vdash \neg M : \mathsf{bool}]\!] = \lambda\gamma\,.\,\lambda i\,.\,\neg\big(s[\![M]\!](\gamma)(i)\big)$$

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- Use the reader monad: $\mathrm{R}\underline{X} := (2 \Rightarrow \underline{X})$:

$$s[\![\diamond \vdash M : \tau]\!] \in \mathrm{R}(s[\![\tau]\!])$$

$$s[\![\Gamma \vdash M : \tau]\!] : s[\![\Gamma]\!] \to \mathrm{R}s[\![\tau]\!]$$

$s[\![M]\!](i)$ = value M returns when i in the cell

$(\mathrm{Fin}, \mathrm{R}, s)$ **is not fully abstract** [Matache & Staton]

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M, M' : ((1 → bool) → bool) → bool

apply f : (1 → bool) → bool to λx . tt
then to λx . ff
then take the disjunction
M := λf . (f (λx . tt)) ∨ (f (λx . ff))

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Intuitively, $M \simeq_{\mathrm{ctx}} M'$. But...

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Have:

$$s[\![M]\!], s[\![M']\!] \in \mathrm{R}\Big(\big((1 \Rightarrow \mathrm{R2}) \Rightarrow \mathrm{R2}\big) \Rightarrow \mathrm{R2}\Big)$$

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Have:

$$s[\![M]\!], s[\![M']\!] \in \mathrm{R}\big(((1 \Rightarrow \mathrm{R2}) \Rightarrow \mathrm{R2}) \Rightarrow \mathrm{R2}\big)$$

Take $\kappa : (1 \Rightarrow \mathrm{R2}) \to \mathrm{R2}$:

$$\kappa(g) := \begin{cases} \mathrm{const}_1 & \text{if } g(*) = \mathrm{const}_1 \\ \mathrm{const}_0 & \text{else} \end{cases}$$

Then

$$s[\![M]\!](i)(\kappa)(j) = 1 \neq 0 = s[\![M']\!](i)(\kappa)(j)$$

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κ knows how g behaves
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State is read-only
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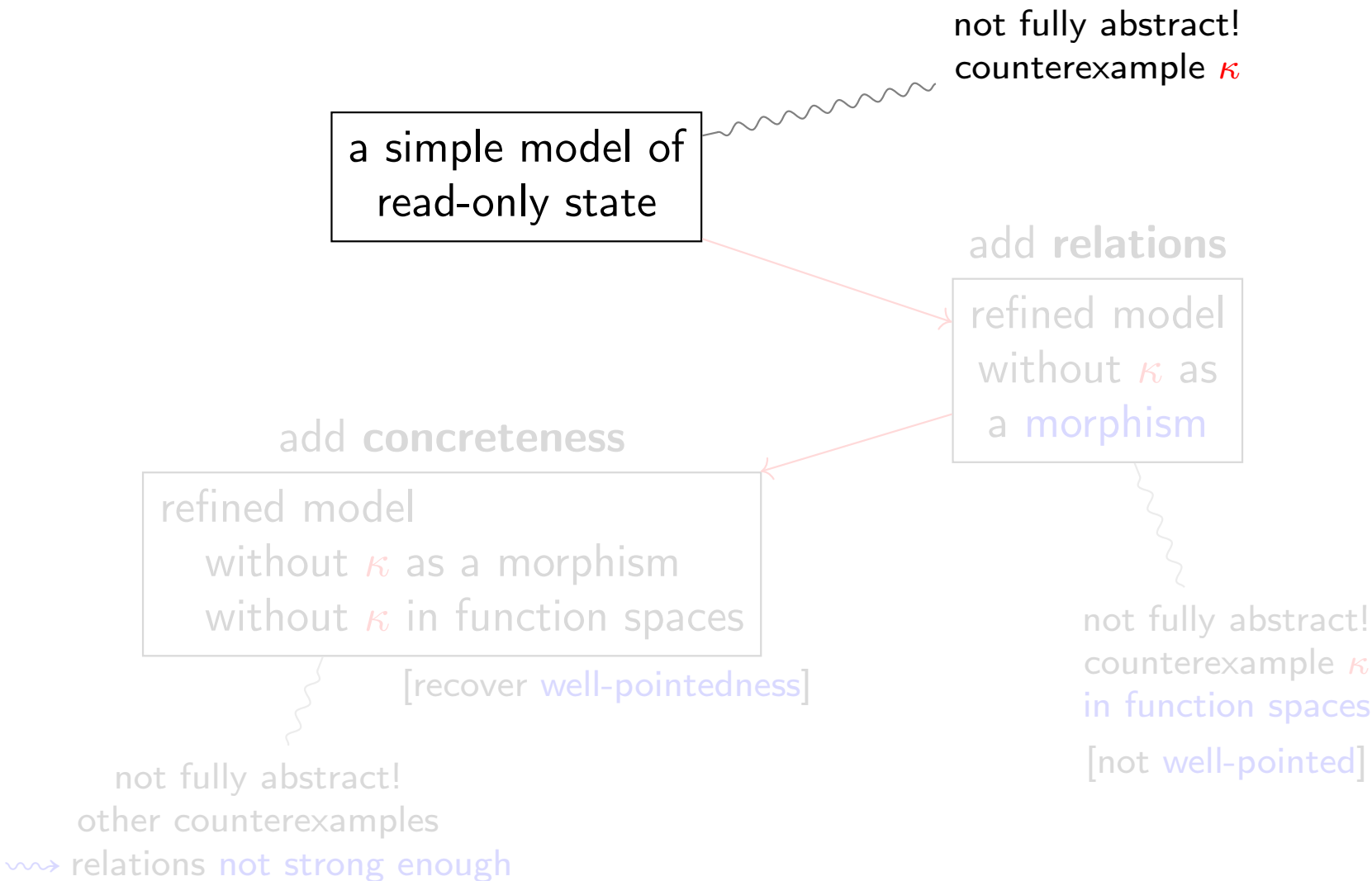
κ is a counterexample to contextual equivalence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counterexample κ
a simple model of
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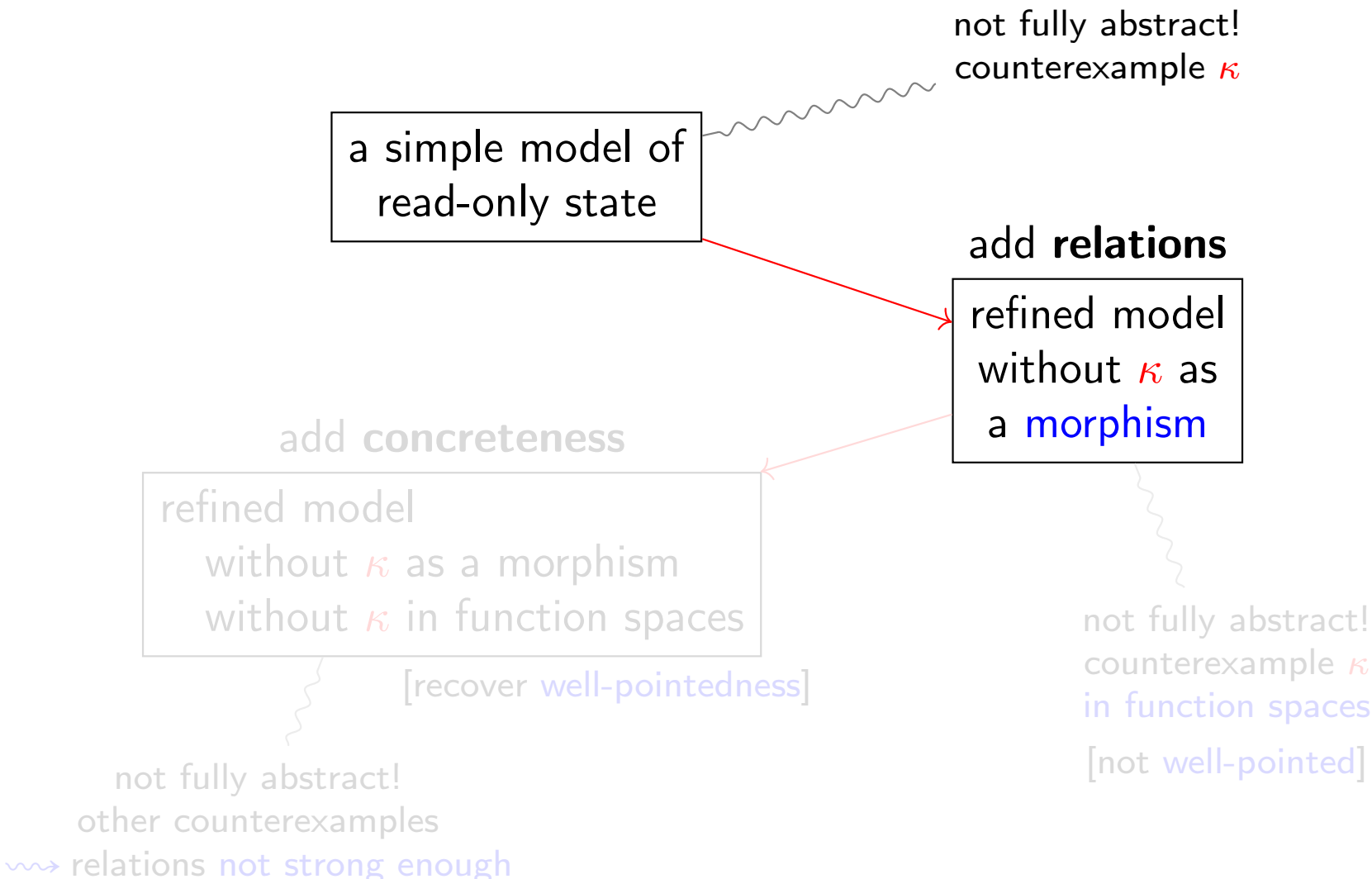
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A refined model $(\mathbb{L}, \hat{\mathrm{R}}, t)$

Idea:

- pair each set with relations R_0 and R_1
- restrict to functions preserving these relations

preserving $R_i \rightsquigarrow$ respecting behaviour when cell contains i

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The cartesian closed category $\mathbb{L}$

objects: triples $(\underline{X}, R_0, R_1)$ $\rightsquigarrow$ $\underline{X} \in \mathsf{Fin}$, $R_i \subseteq \underline{X}^2$

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The cartesian closed category $\mathbb{L}$

objects: triples $(\underline{X}, R_0, R_1)$ $\rightsquigarrow$ $\underline{X} \in \mathsf{Fin}$, $R_i \subseteq \underline{X}^2$

maps $(\underline{X}, R_0, R_1) \to (\underline{Y}, S_0, S_1)$:

maps $f : \underline{X} \to \underline{Y}$ preserving the relations

$\rightsquigarrow$ $(x, x') \in R_i \implies (f\,x, f\,x') \in S_i$ for $i = 1, 2$

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The monad $\hat{\mathrm{R}}$ $\rightsquigarrow$ defined by $\top\top$-lifting

$$\hat{\mathrm{R}}(\underline{Y}, S_0, S_1) = \big(2 \Rightarrow \underline{Y}, \hat{\mathrm{R}}(S_0), \hat{\mathrm{R}}(S_1)\big)$$

$(h, h') \in \hat{\mathrm{R}}(S_i)$
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$(h, h') \in \hat{\mathrm{R}}(S_i)$
$\iff (h\, i, h'\, i) \in S_i$

The interpretation t

$t(\mathsf{bool}) = \big(2, \{(0,0),(1,1)\}, \{(0,0),(1,1)\}\big)$
$t(\mathrm{read}) = s[\![\mathrm{read}]\!]$ ⇝ already preserves the relations
...

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κ is not a morphism in $\mathbb{L}$!

$\Downarrow$

we've removed a
counterexample
to contextual equivalence

$(\mathbb{L}, \hat{\mathrm{R}}, t)$ — κ removed

$\downarrow U$ forgetful

$(\mathrm{Fin}, \mathrm{R}, s)$ — κ lives here

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All the structure is preserved:

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$\Longrightarrow$ . . . so the semantics is preserved:

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This can **never** be enough $\rightsquigarrow$ relations are never sufficient

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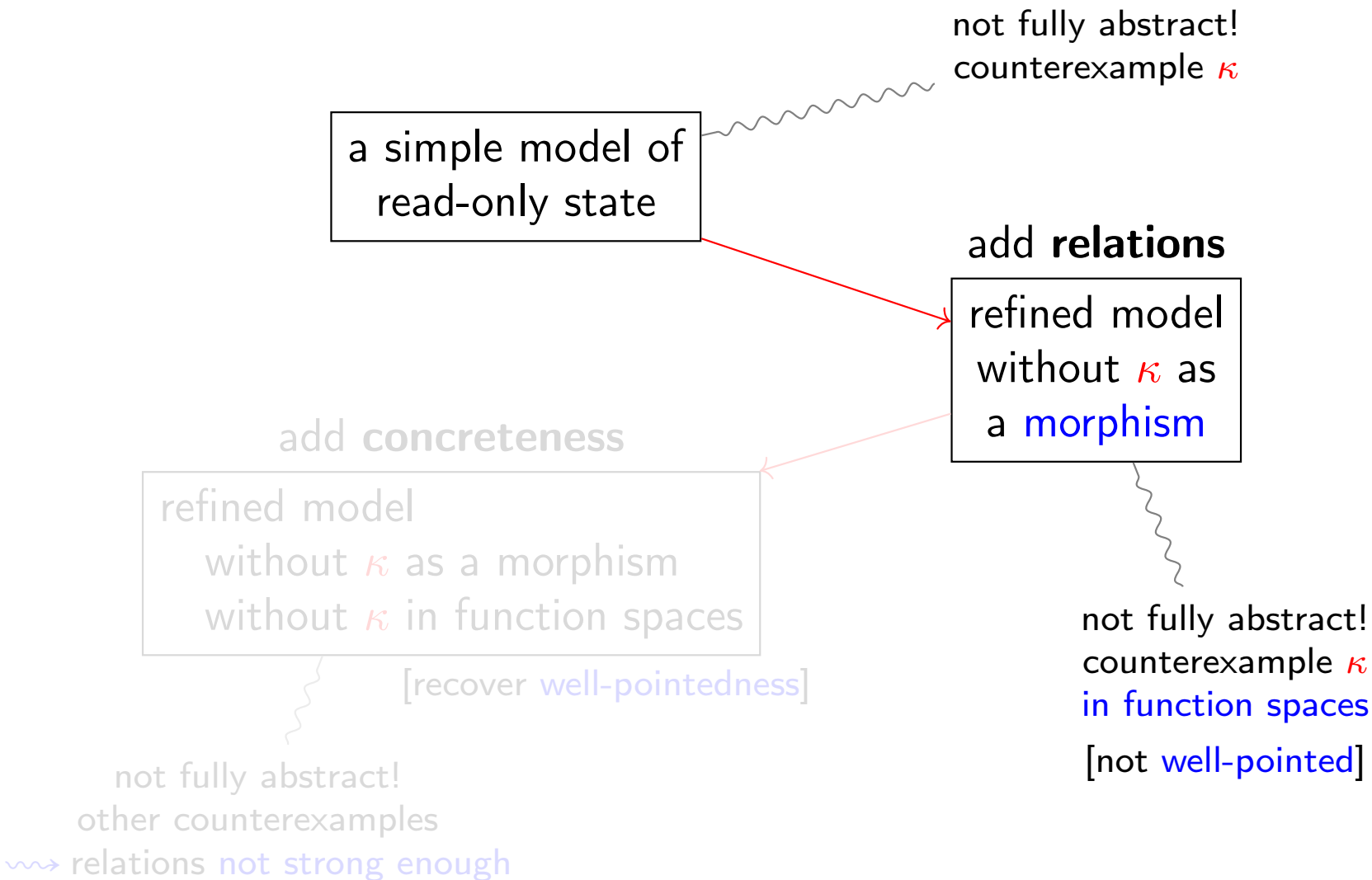
κ is not in the hom-sets in $\mathbb{L}$ **but** κ is in the function spaces in $\mathbb{L}$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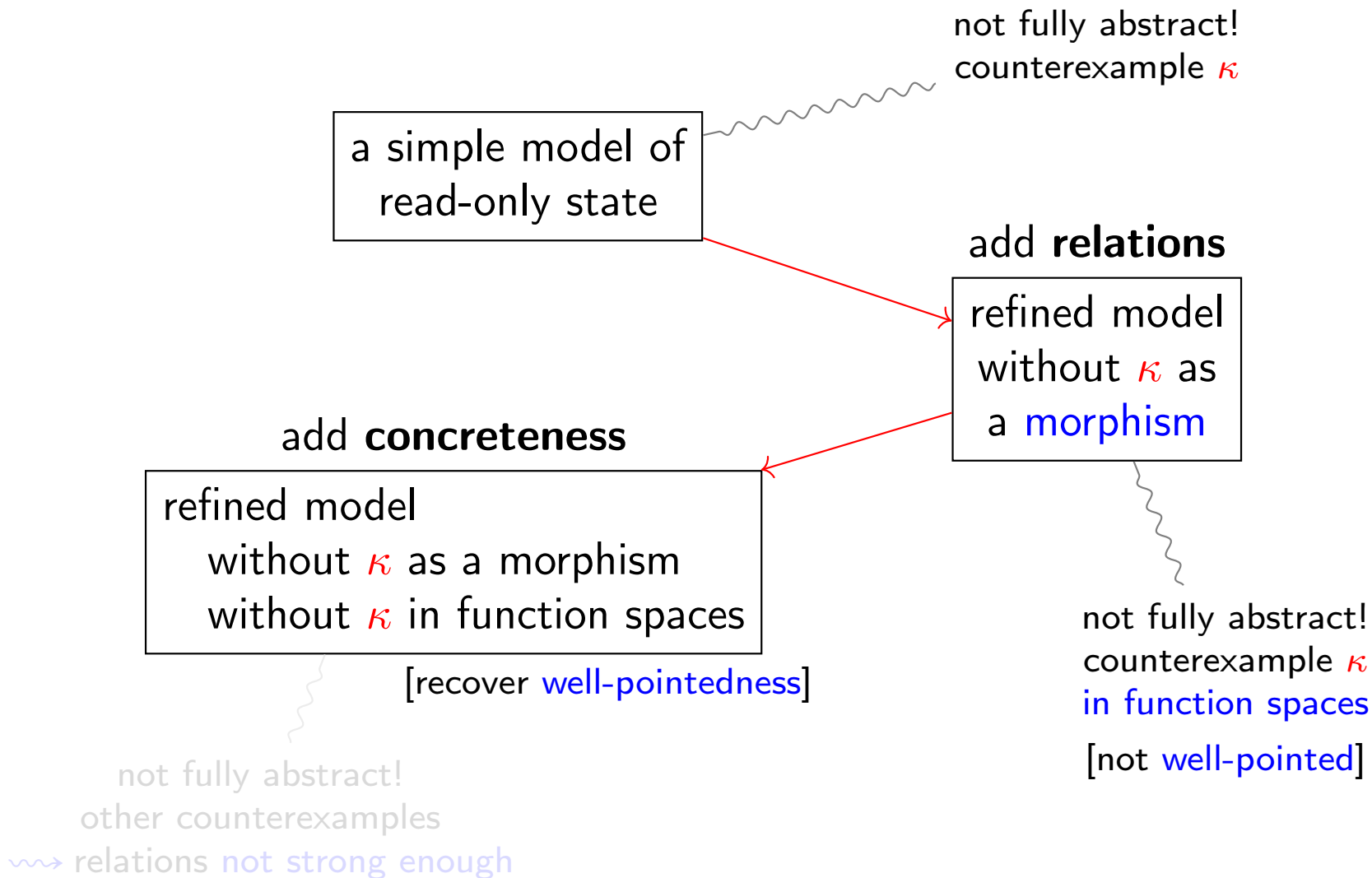
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Removing κ from the function space

Removing κ from the function space

Observation

We have

$$\kappa \in U(t[\![(1 \to \mathrm{bool}) \to \mathrm{bool}]\!])$$

but there is no global element in $\mathbb{L}$

$$\gamma : 1 \to t[\![(1 \to \mathrm{bool}) \to \mathrm{bool}]\!]$$

such that $\gamma(*) = \kappa$.

$\Rightarrow \mathbb{L}$ is not well-pointed

[$f = g$ iff $f \circ \gamma = g \circ \gamma$
for all global elements γ]

Removing κ from the function space

Observation $\rightsquigarrow$ κ appears as a shadow!

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Solution: restrict to things named by a global element

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Explicitly: for every $x \in \underline{X}$, the pair $(x, x) \in R_0$ and $(x, x) \in R_1$.

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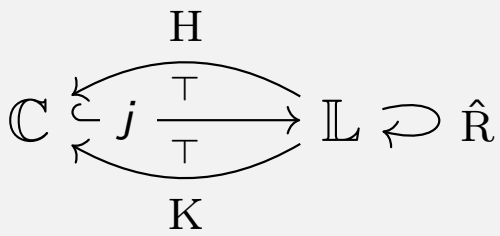

union relations
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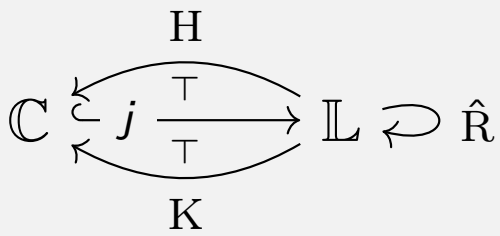

union relations
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$\Rightarrow$

$\mathbb{C}$ a CCC
with monad $\mathrm{H}\hat{\mathrm{R}}j$

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objects: $(\underline{X}, R_0, R_1)$ such that $x \in \underline{X} \implies (x, x) \in R_i$

maps: set-maps preserving the relations

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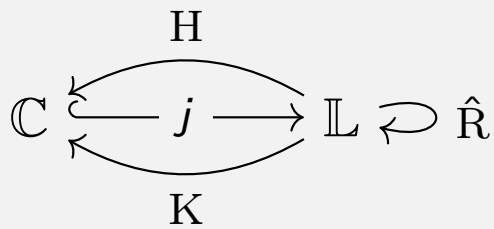

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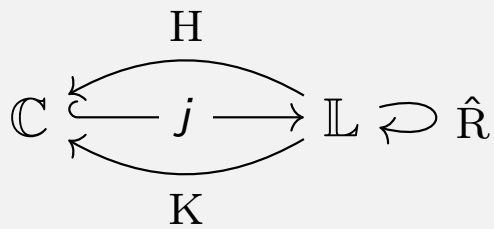

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Exponentials internalise preserving relations

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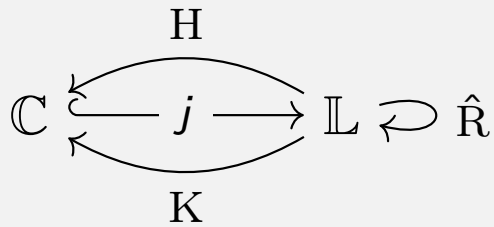

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iff $f \in U\big(X \Rightarrow_{\mathbb{L}} Y\big)$

and $\exists$ a global element in $\mathbb{L}$ corresponding to f

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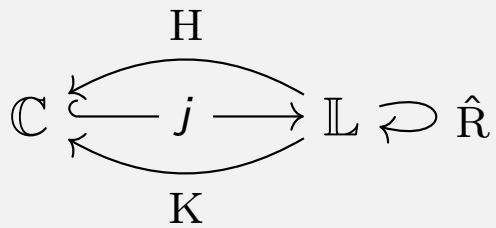

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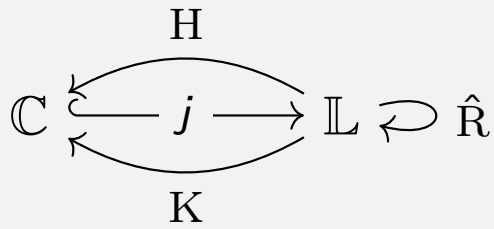

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$\rightsquigarrow$ concreteness removes κ from the function space

Success!

Our new semantic model:

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Aim: remove bad morphism κ

1. from hom-sets $\rightsquigarrow$ [logical] relations
2. from function spaces $\rightsquigarrow$ concreteness

Success?

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Aim: remove bad morphism κ

1. from hom-sets $\rightsquigarrow$ [logical] relations
2. from function spaces $\rightsquigarrow$ concreteness

But $(\mathbb{C}, \mathrm{H}\hat{\mathrm{R}}j, t)$ is not fully abstract $\rightsquigarrow$ need stronger relations!

We know how to remove a single bad morphism...

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Question:

how can we soup up $\mathbb{C}$ to remove every bad morphism?

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A sufficient condition:

$f = s[\![K]\!]$ for some K $\rightsquigarrow$ f is definable

The route from here

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1. the model we've just seen, abstractly

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4. getting full abstraction

Constructing $(\mathbb{C}, \mathrm{H}\hat{\mathrm{R}}j, t)$: a recipe

A recipe for removing bad morphisms κ

1. Use relations $\rightsquigarrow$ stops them being morphisms
2. Use concreteness $\rightsquigarrow$ cuts them out function spaces

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$$\begin{array}{c} \mathrm{Fin} \\ \circlearrowleft \\ \mathrm{R} \end{array}$$

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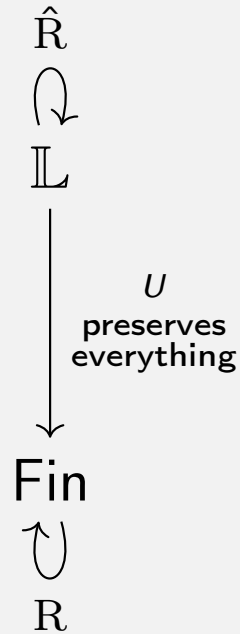

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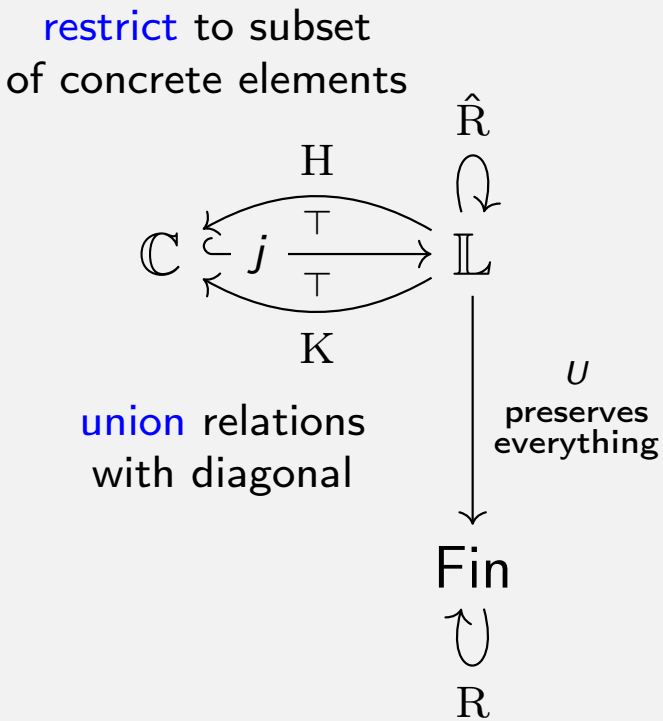

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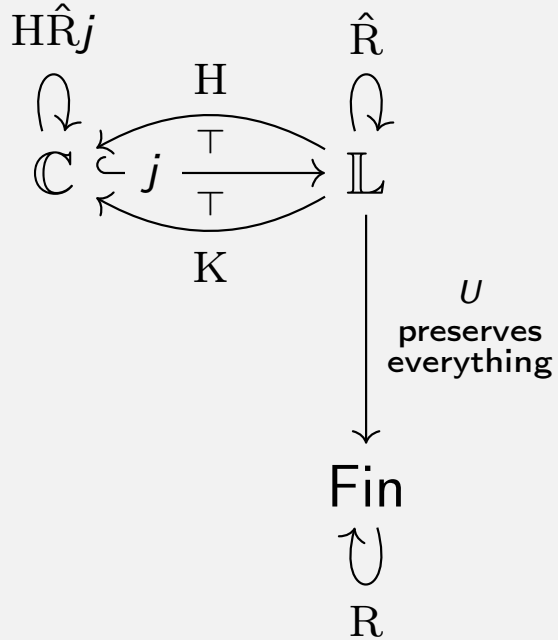

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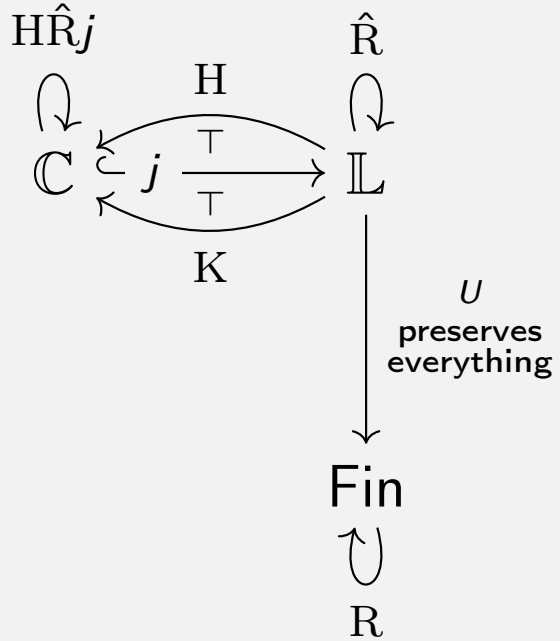

(X, A) such that $A \subseteq X$;
maps preserve the predicate

$$\mathrm{Sub}(\mathsf{Fin}) \times \mathrm{Sub}(\mathsf{Fin})$$

$$\downarrow \; \mathrm{cod} \times \mathrm{cod}$$

$$\mathrm{cod} : (A \subseteq X) \mapsto X$$

$$\mathsf{Fin} \times \mathsf{Fin}$$

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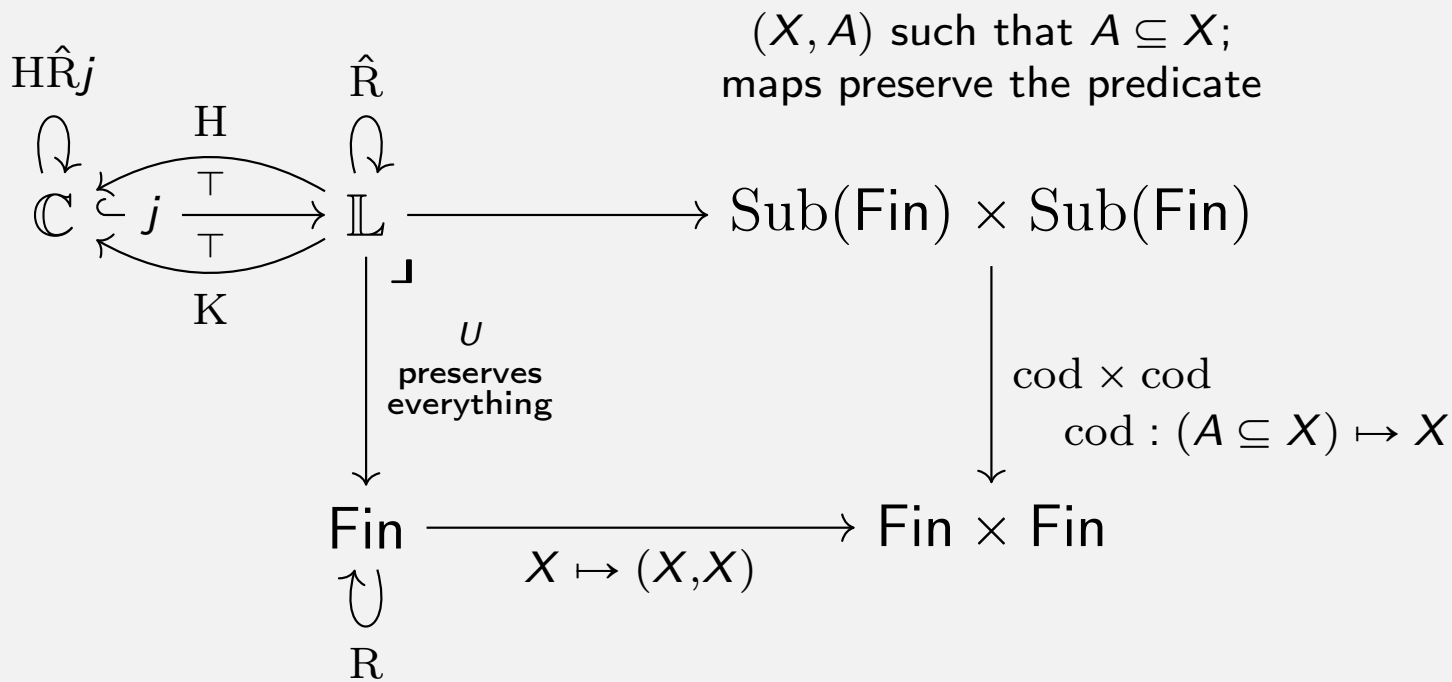

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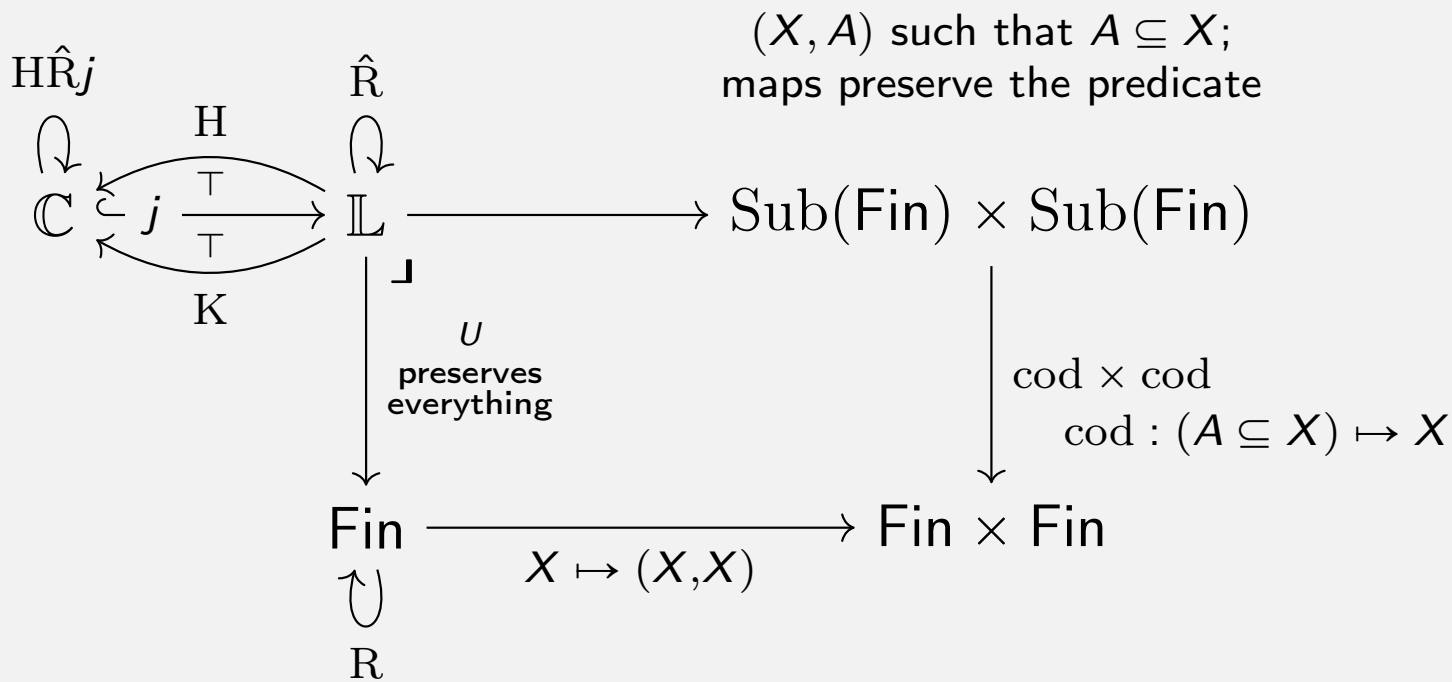
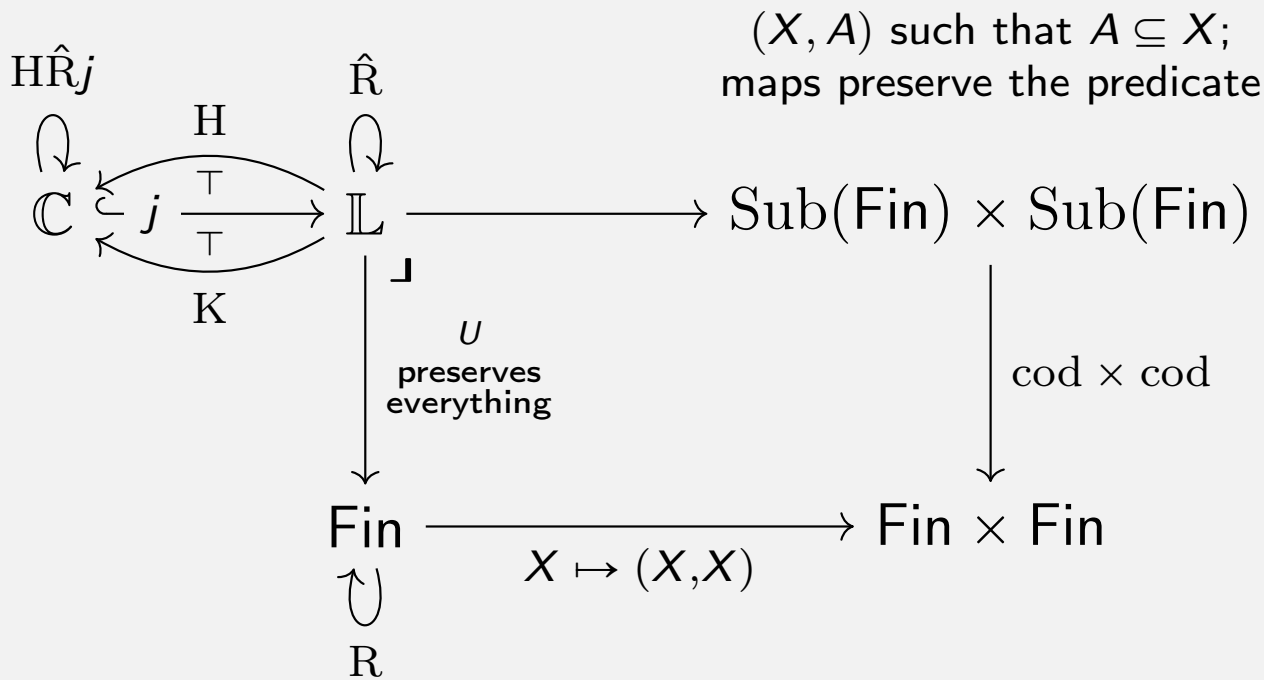


NB: two relations $\rightsquigarrow$ two categories on RHS

A recipe for removing bad morphisms κ



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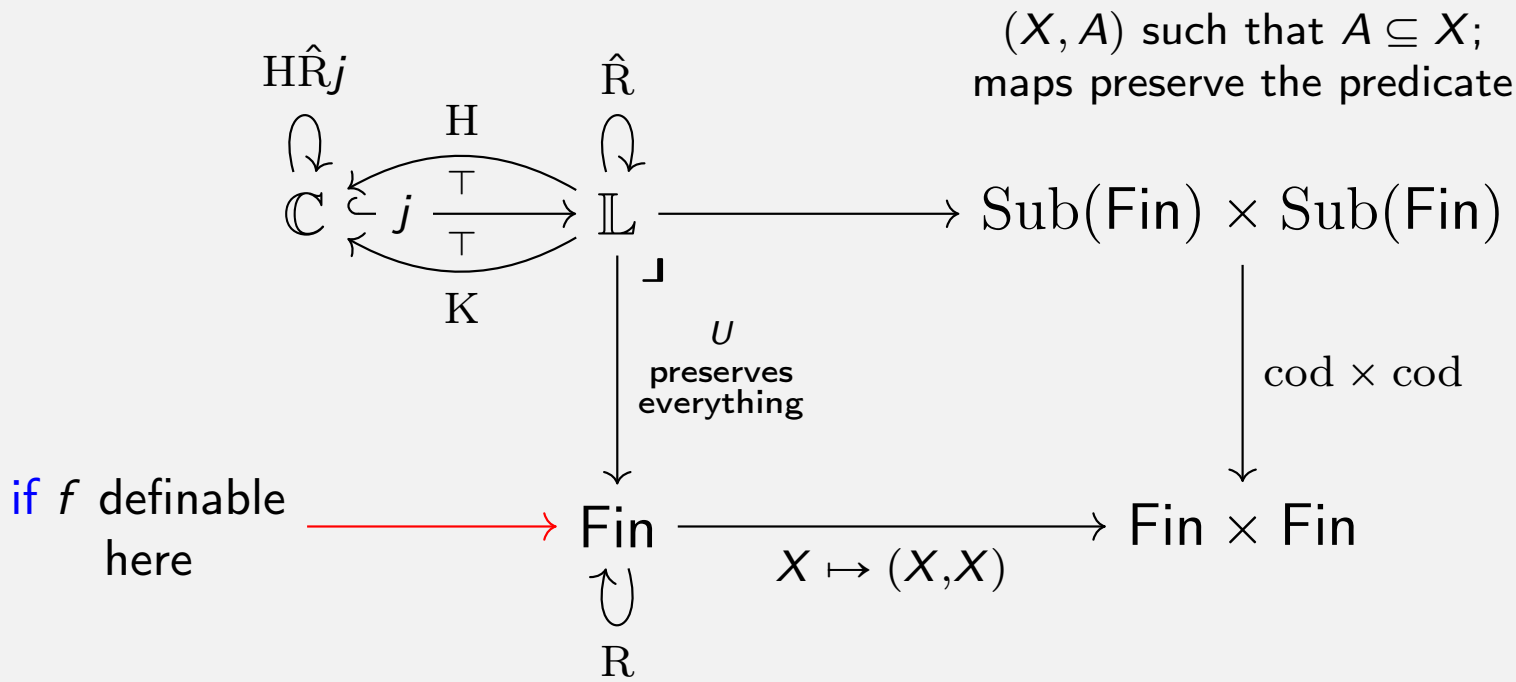

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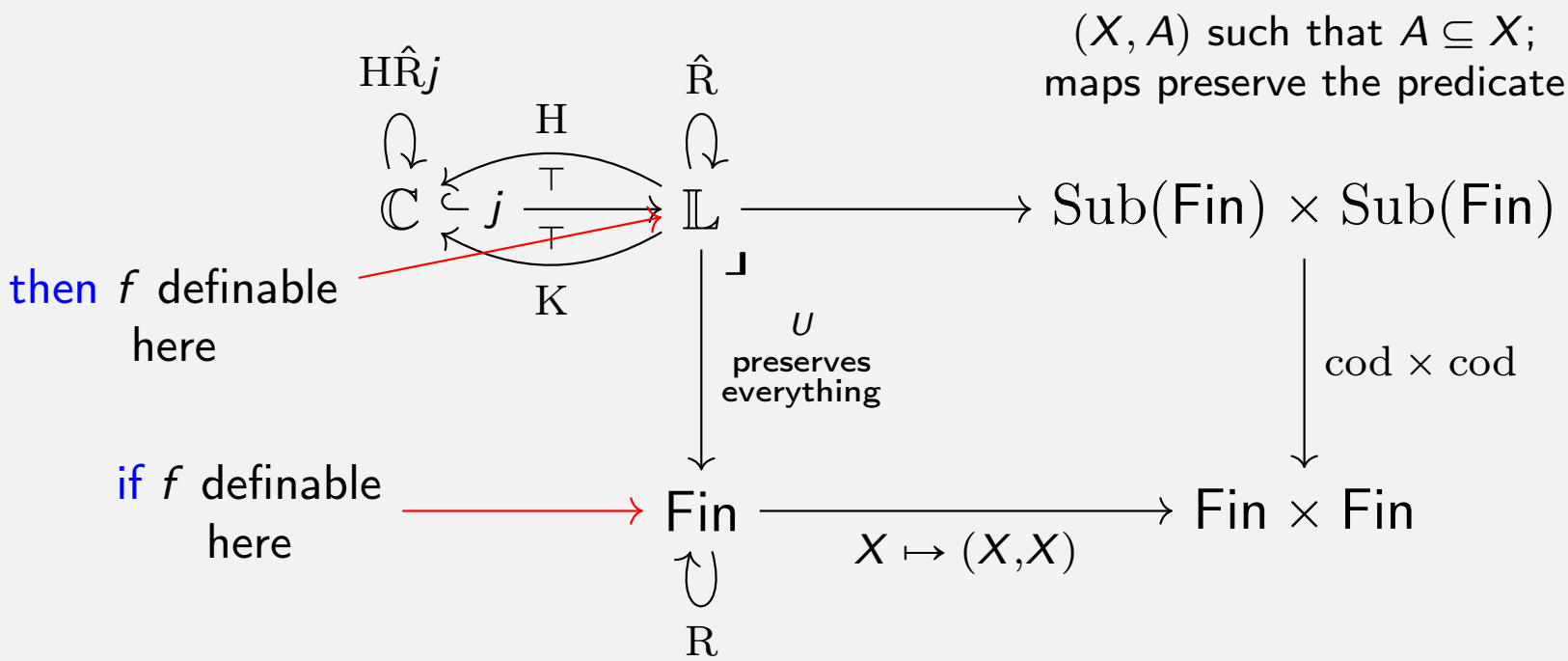
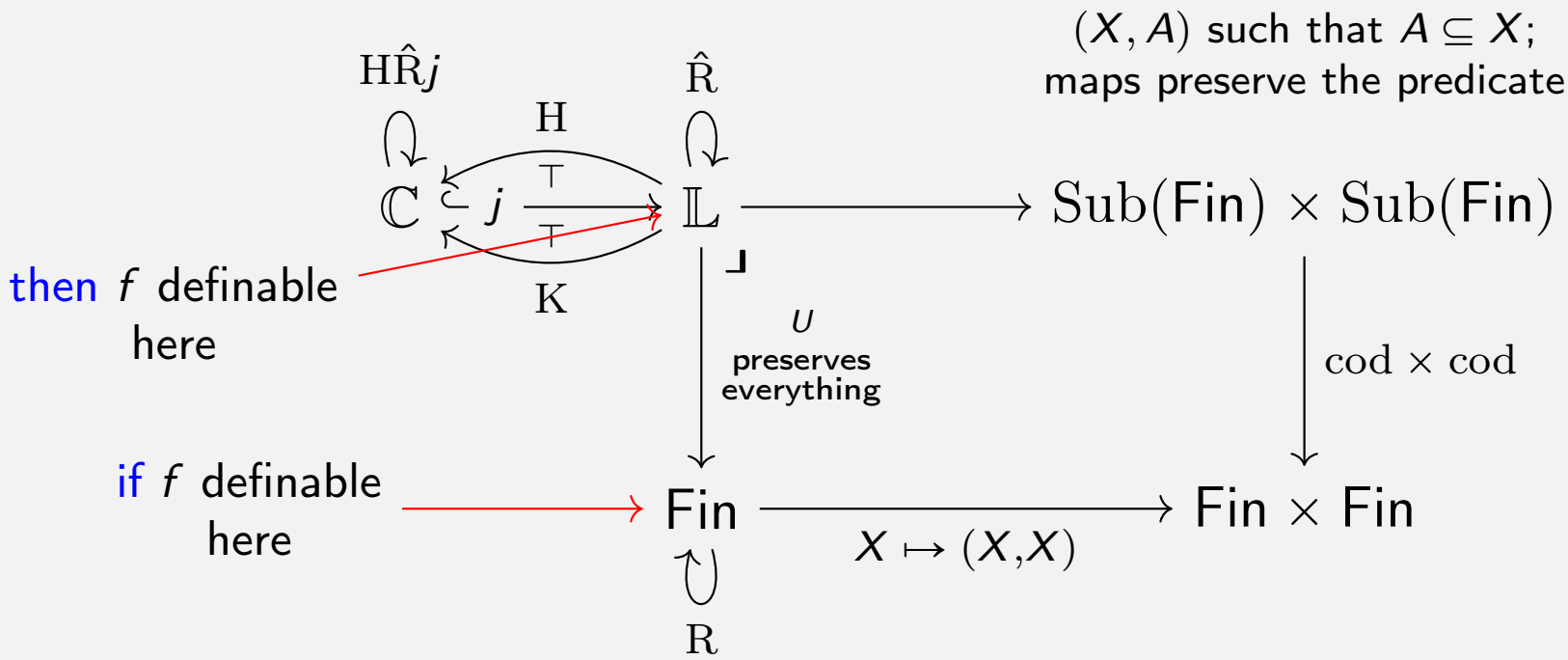


A recipe for removing bad morphisms κ



Every definable $f : s[\![\Gamma]\!] \to T(s[\![\sigma]\!])$ lifts to $f : t[\![\Gamma]\!] \to \hat{T}(t[\![\sigma]\!])$

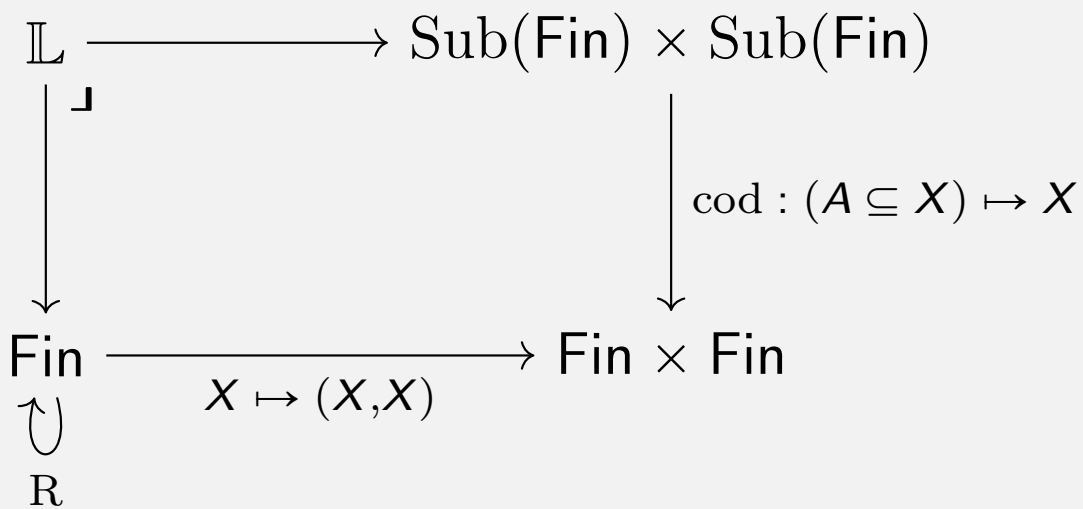
How can we change this construction so only definable maps lift?

$\mathbb{K}_{\mathcal{M},F}$, abstractly

Idea:

replace relations on sets . . .

(X, A) such that $A \subseteq X$;
maps preserve the predicate

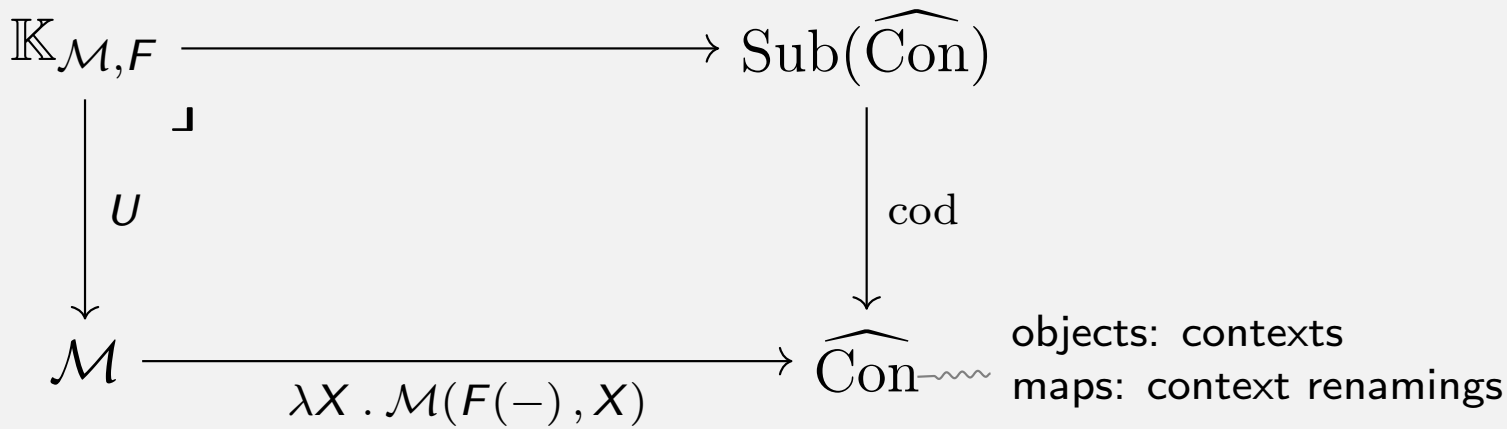


$\mathbb{K}_{\mathcal{M},F}$, abstractly

Idea:

replace relations on sets . . .

with families of relations indexed by contexts



The category $\mathbb{K}_{\mathcal{M},F}$ of Kripke relations [Jung & Tiuryn, Alimohamed]

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category of contexts and renamings

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$f : W \to W'$

preserves the predicate

$h \in R(\Gamma) \Rightarrow (f \circ h) \in R'(\Gamma)$

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$(f_1, \ldots, f_n) \in (R \supset S)(\Gamma)$ iff, for any $\rho : \Gamma \to \Delta$,

$(x_1, \ldots, x_m) \in R(\Delta) \subseteq W^{F\Gamma} \Rightarrow (f_{\rho(1)}x_1, \ldots, f_{\rho(m)}x_m) \in S(\Delta) \subseteq X^{F\Delta}$

Kripke relations for STLC definability [Jung & Tiuryn, Alimohamed, . . .]

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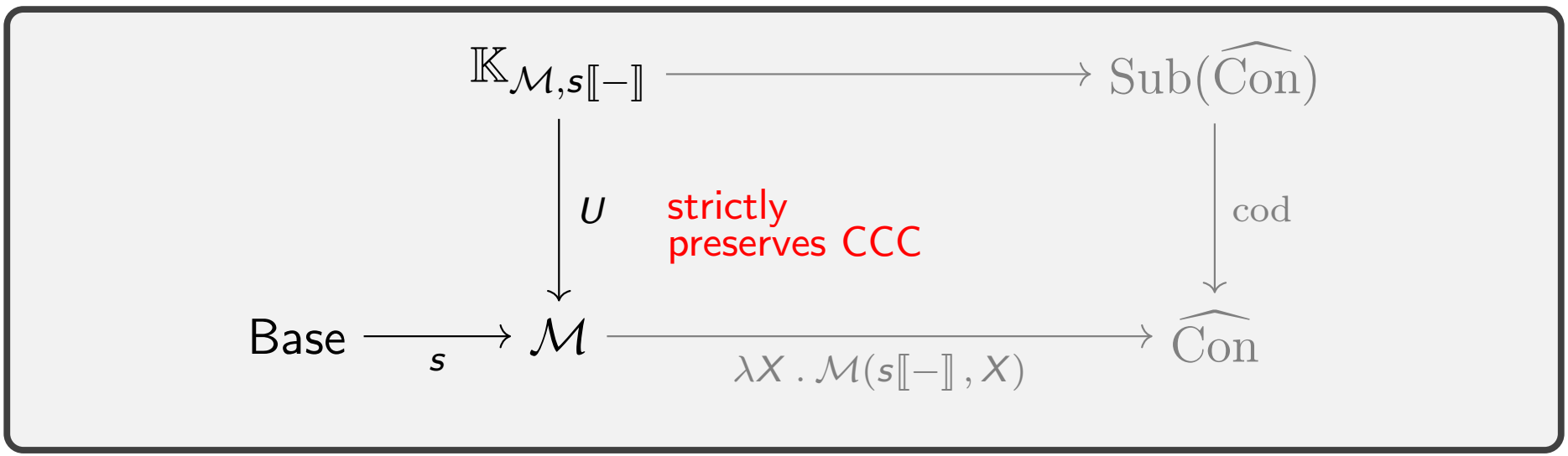
Characterising STLC-definable maps [Jung & Tiuryn, Alimohamed]



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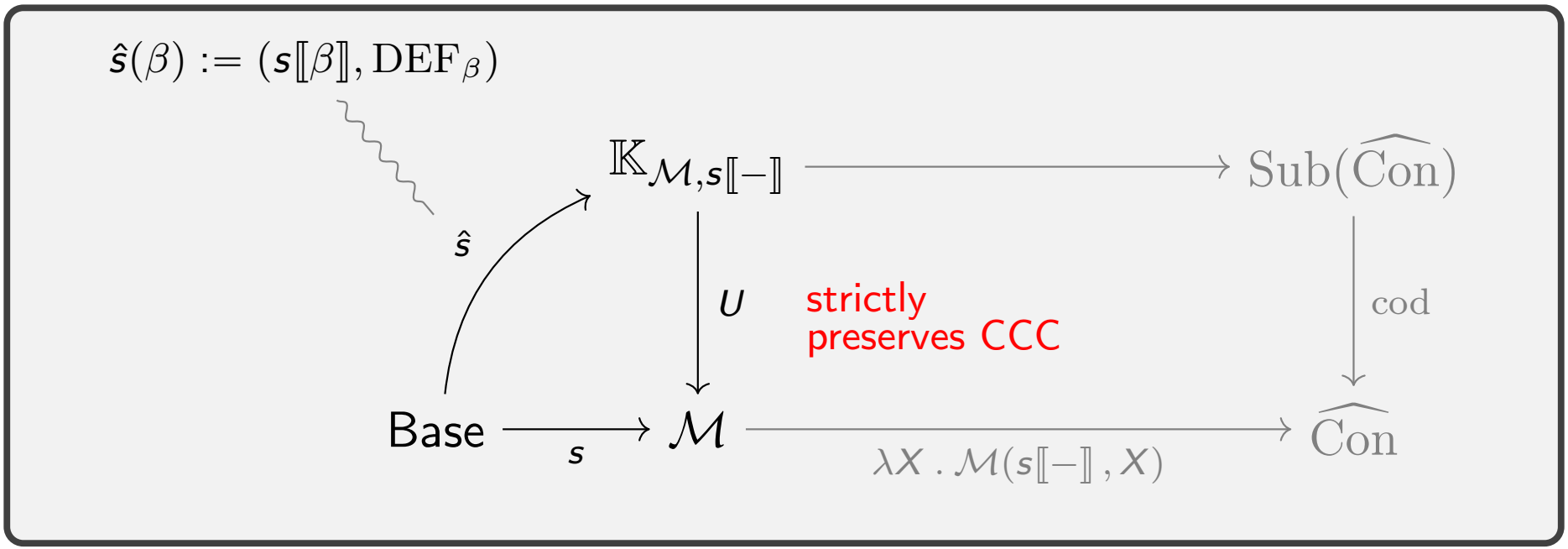

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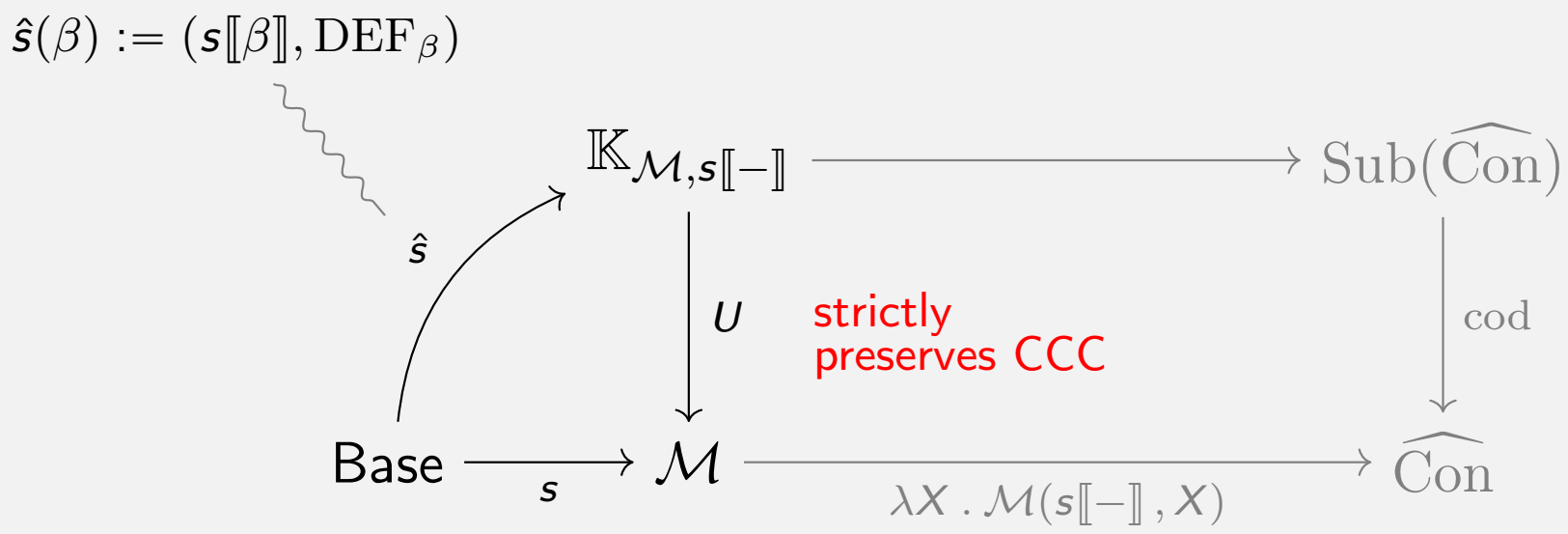

$\hat{s}[\![\sigma]\!] = (s[\![\sigma]\!], \mathrm{DEF}_\sigma)$ for every STLC-type σ.

i.e. $(\mathrm{DEF}_\sigma \supset \mathrm{DEF}_\tau) = \mathrm{DEF}_{\sigma \Rightarrow \tau}$ *etc.*

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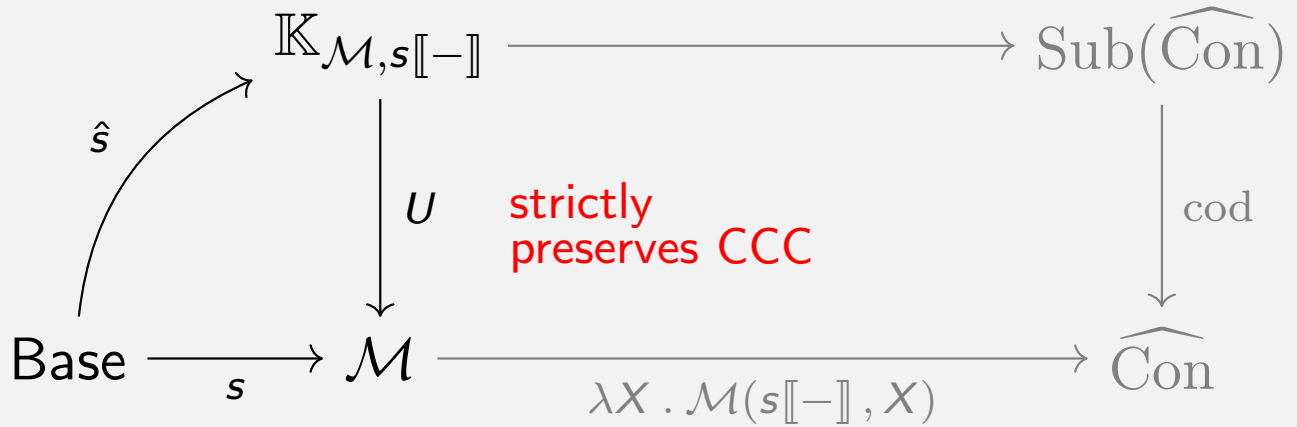

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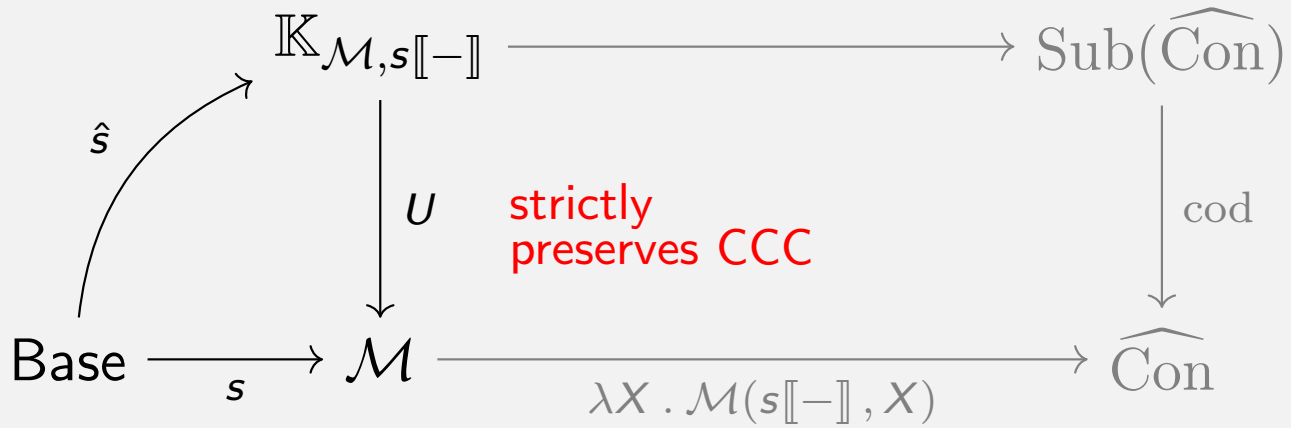

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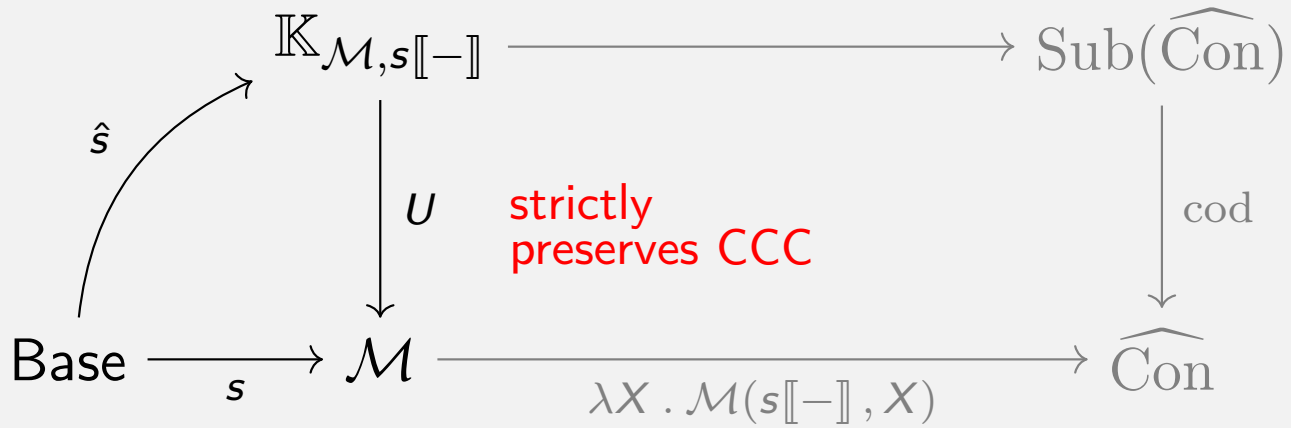

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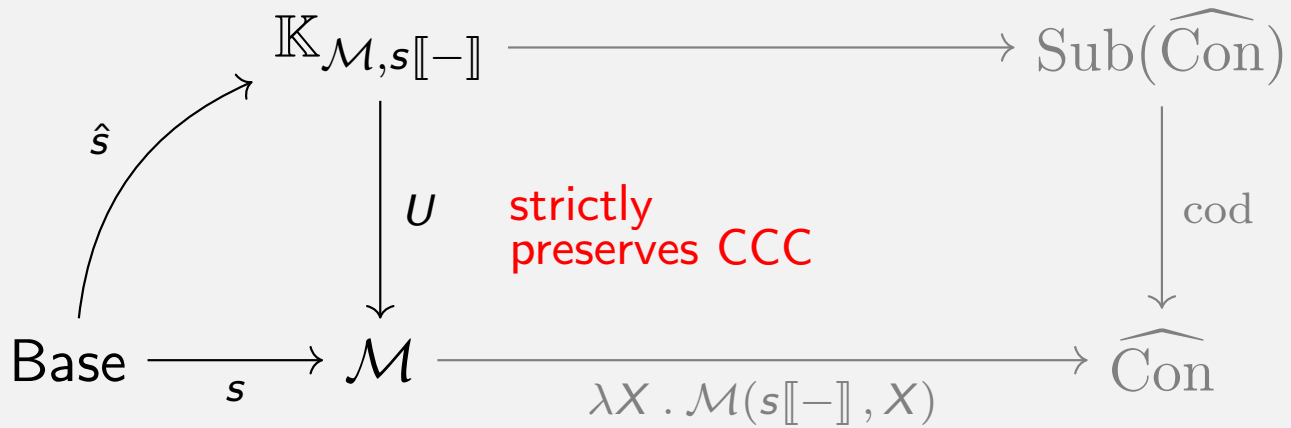

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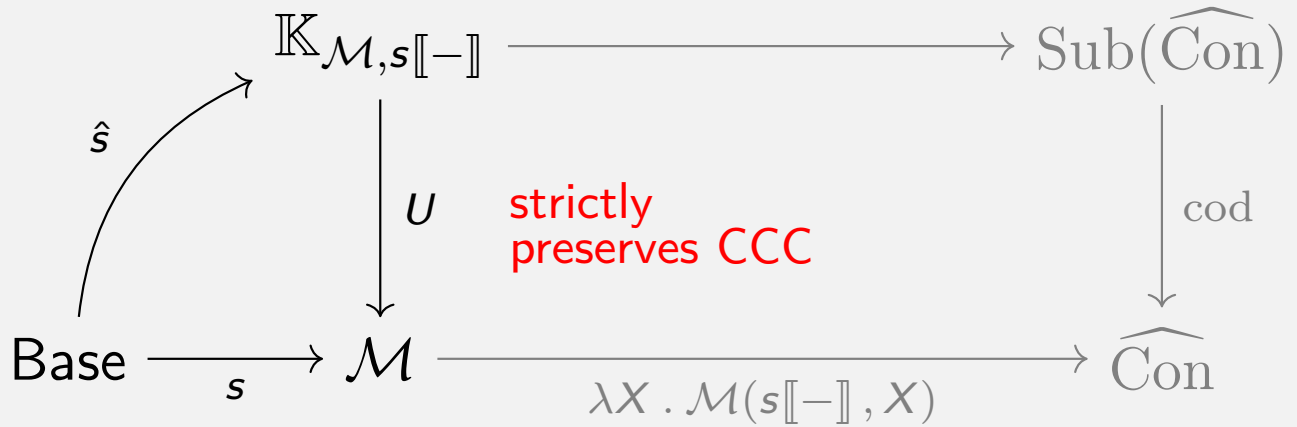

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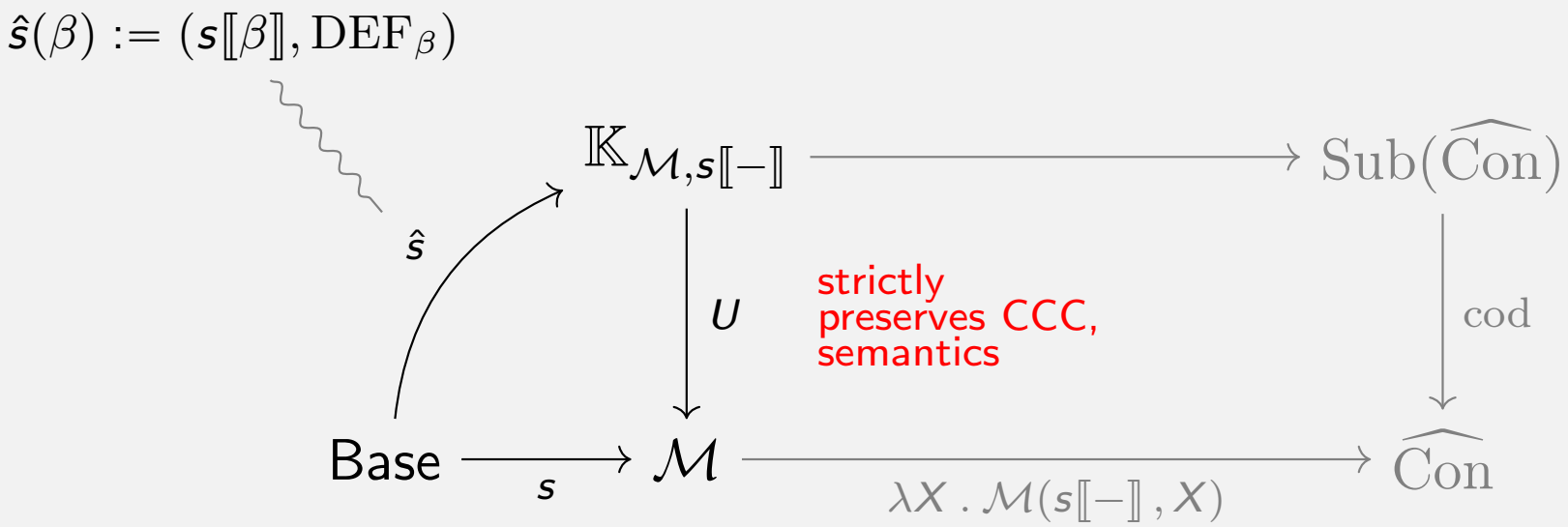
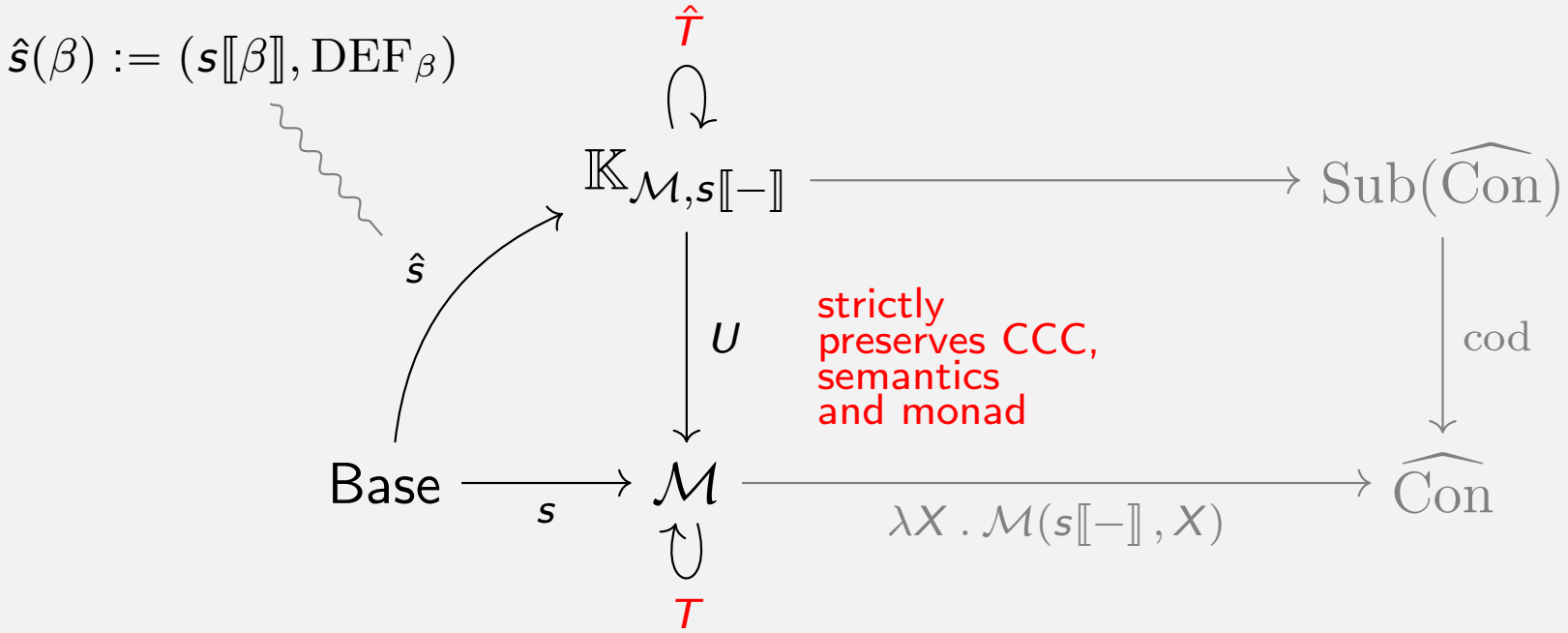
What about the monad?


$\hat{s}(\beta) := (s[\![\beta]\!], \mathrm{DEF}_\beta)$
$\hat{s}$
$\mathbb{K}_{\mathcal{M},s[\![-]\!]}$
$\widehat{\mathrm{Sub}(\mathrm{Con})}$
U
strictly
preserves CCC,
semantics
cod
Base
s
$\mathcal{M}$
$\lambda X \,.\, \mathcal{M}(s[\![-]\!], X)$
$\widehat{\mathrm{Con}}$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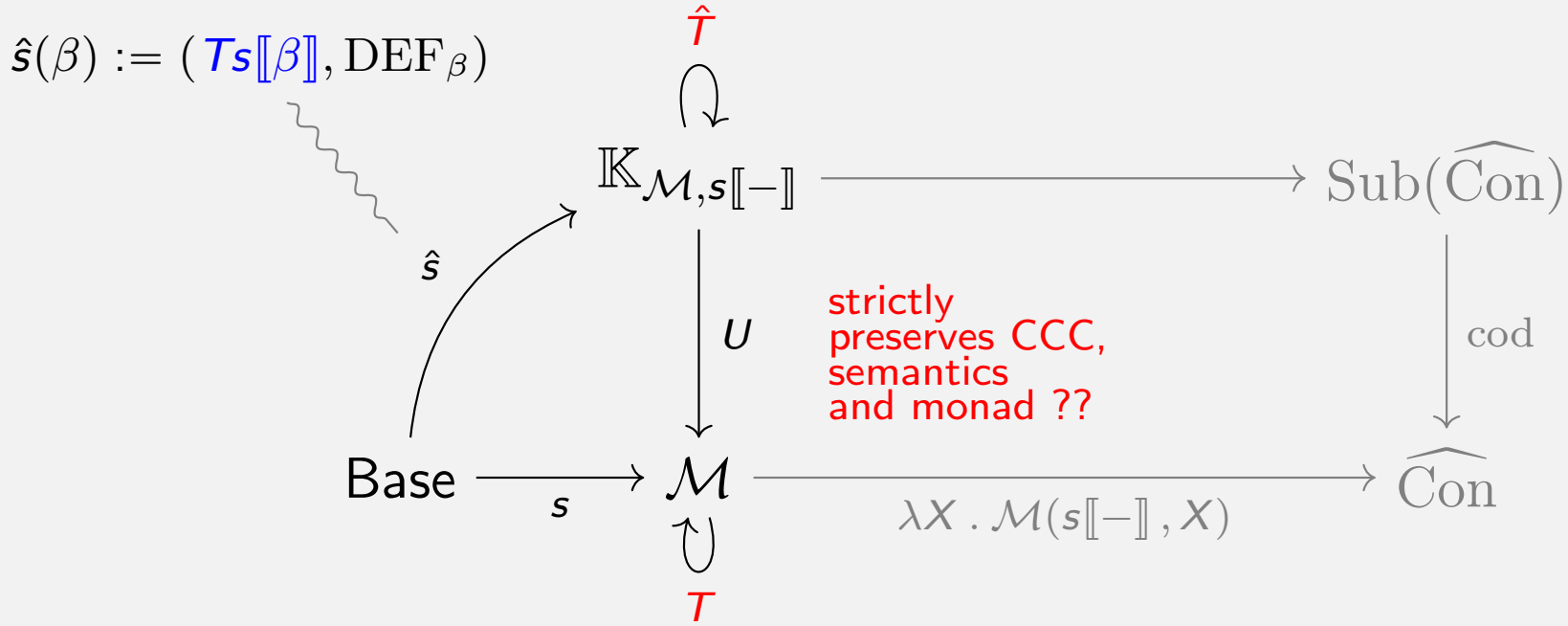
$\hat{s}(\beta) := (Ts[\![\beta]\!], \mathrm{DEF}_\beta)$
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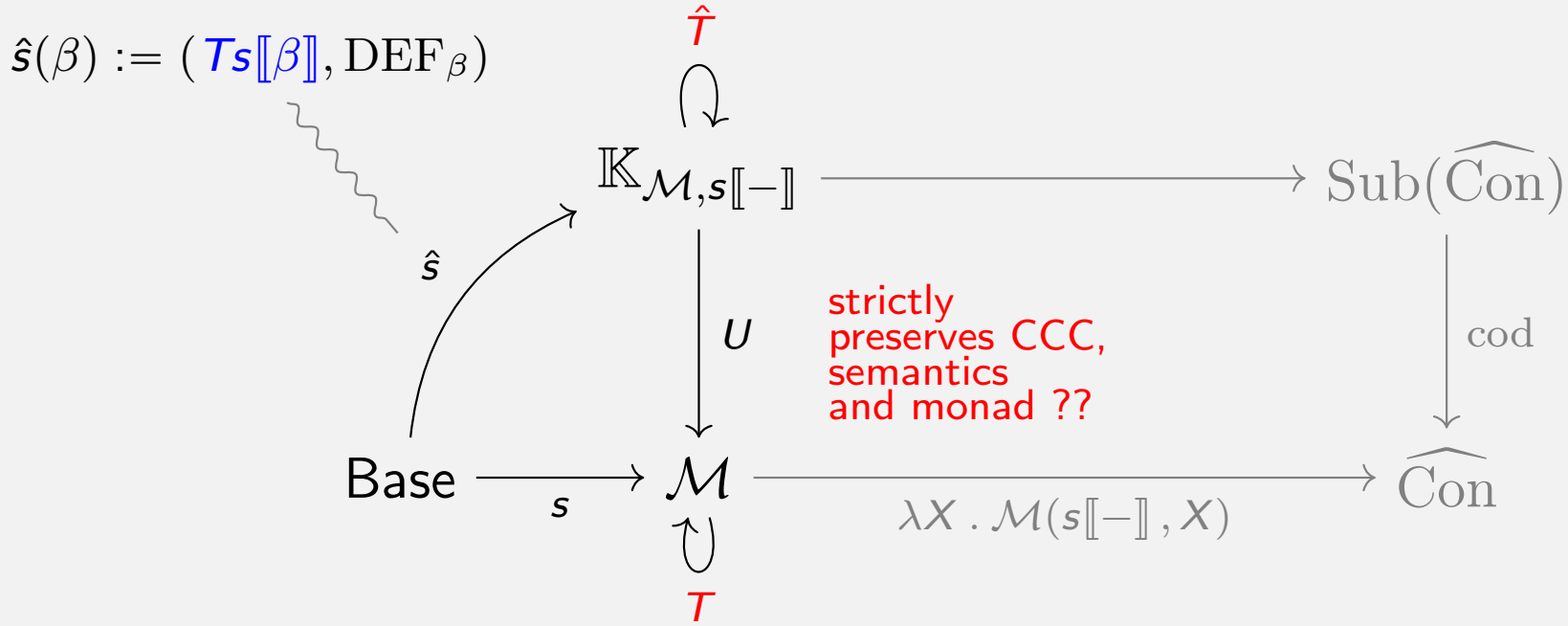
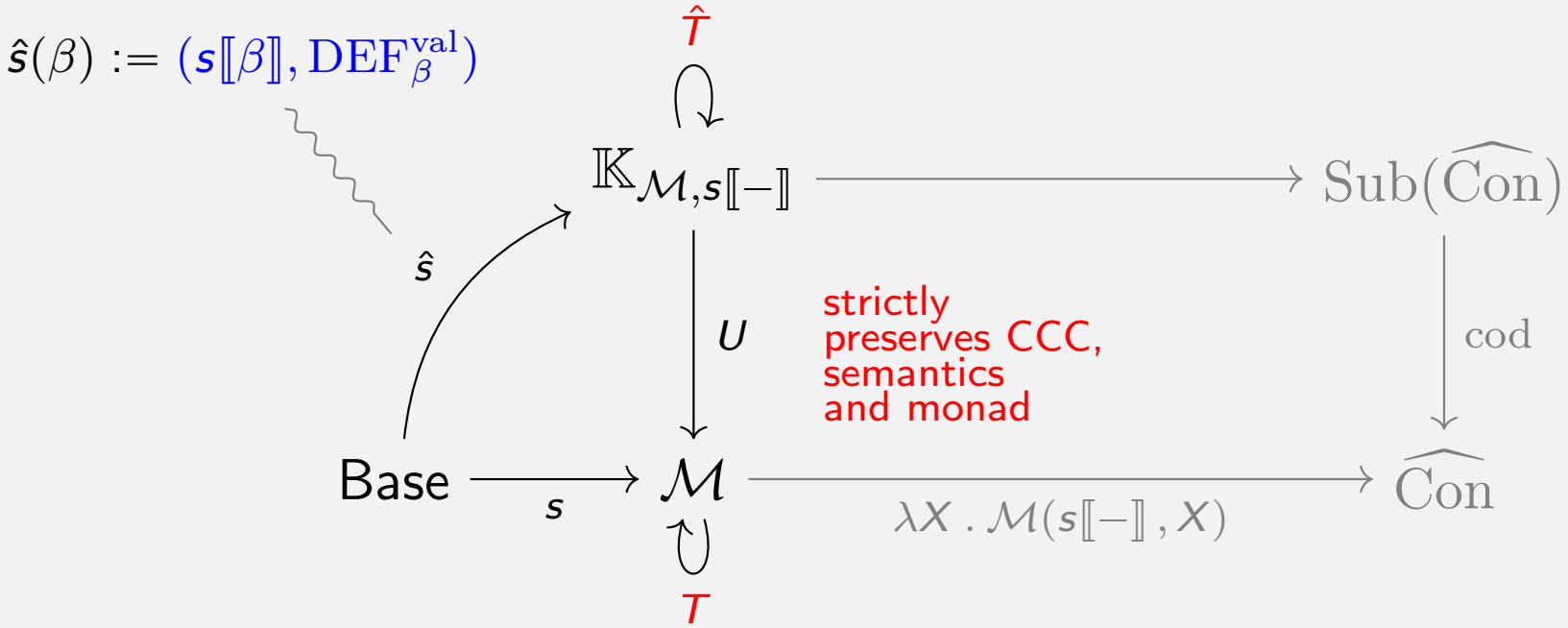


Restricting to values

for $(TX, R) \in \mathbb{K}_{\mathcal{M},F}$

$$R^{\mathrm{val}}(\Gamma) := \left\{ f \;\middle|\; \eta_X \circ f \in R(\Gamma) \right\}$$

so $(X, R^{\mathrm{val}}) \in \mathbb{K}_{\mathcal{M},F}$



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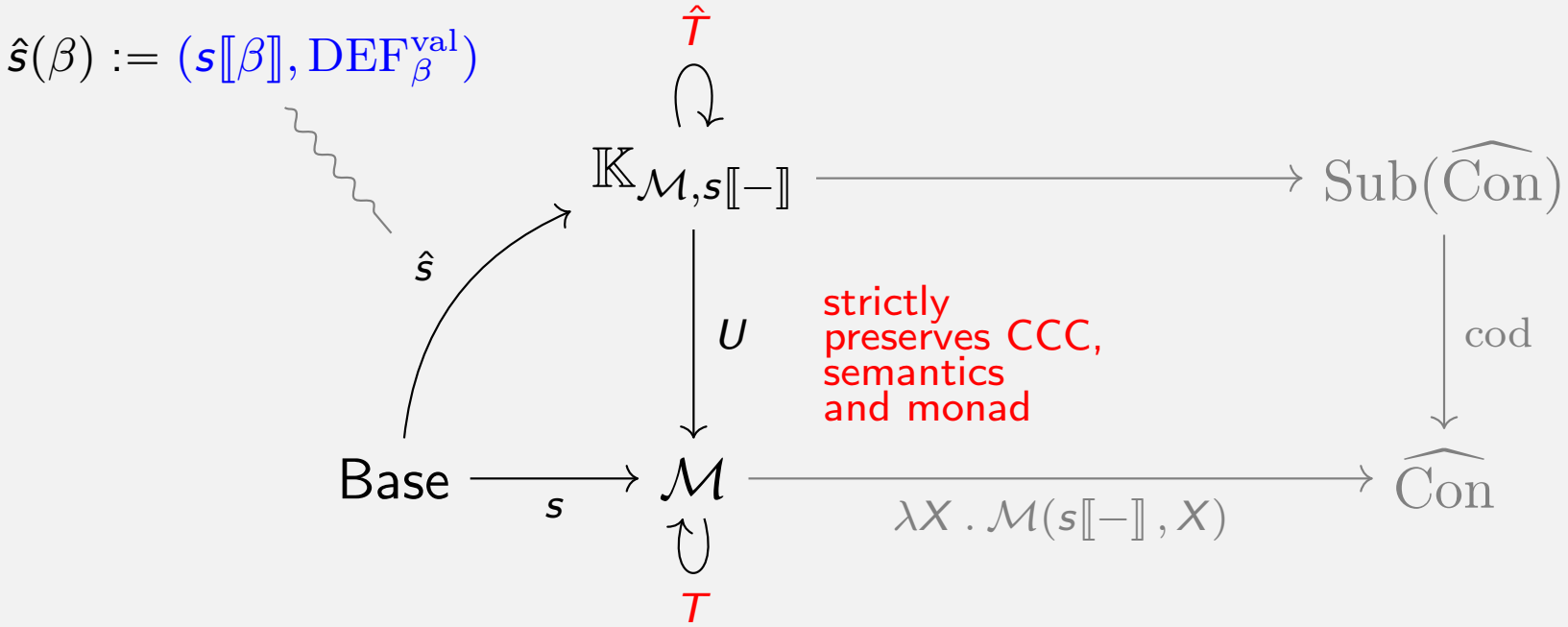
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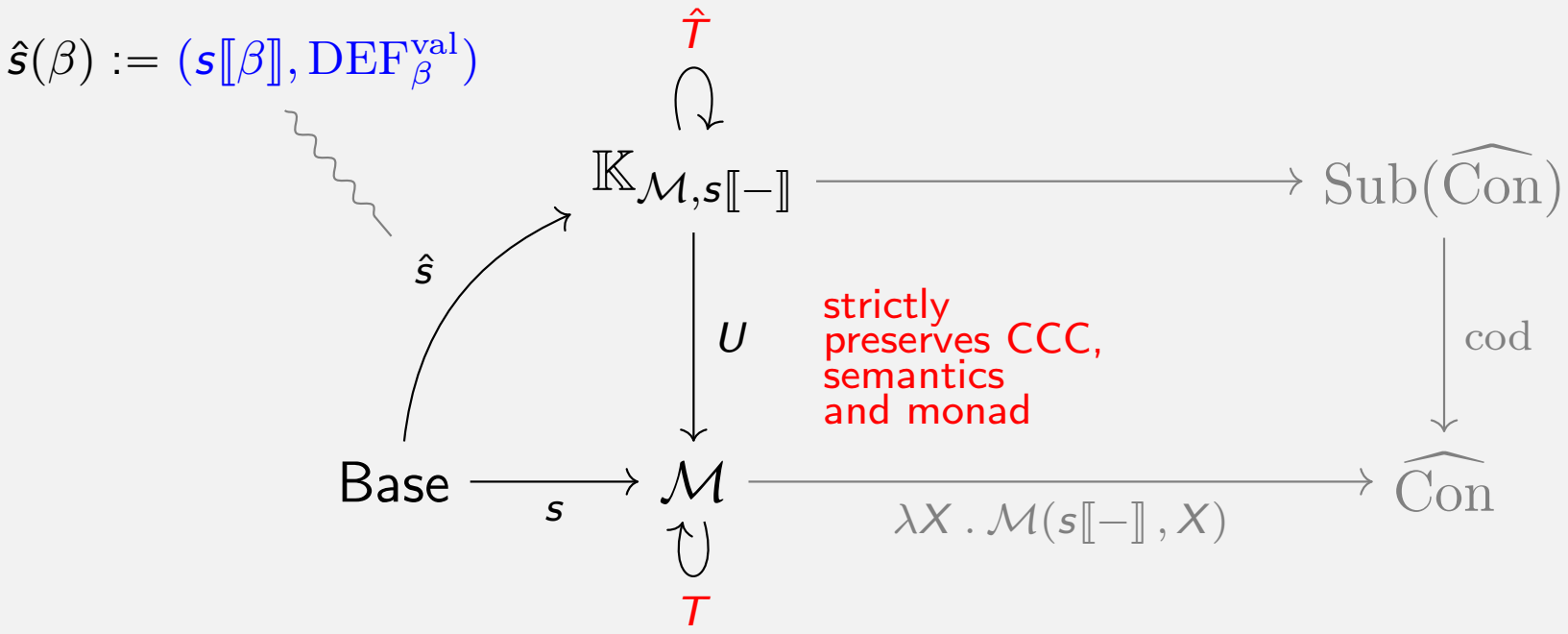
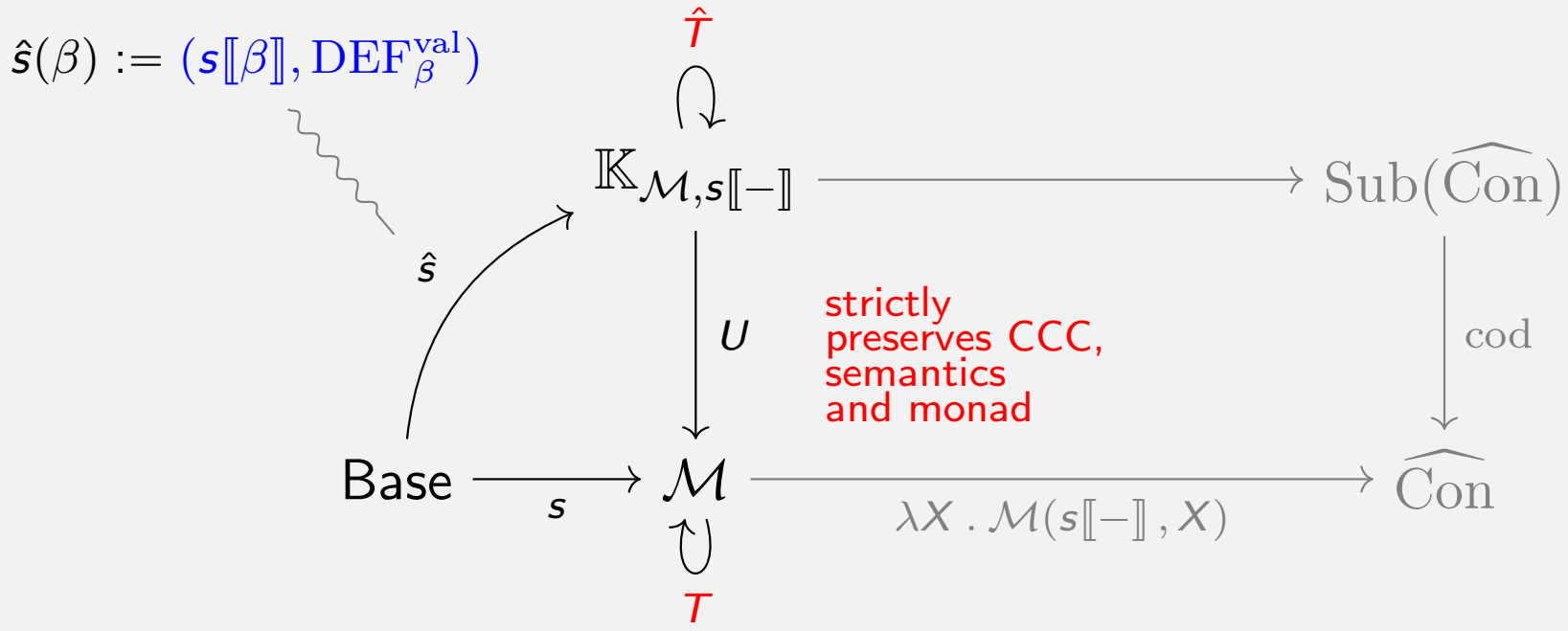
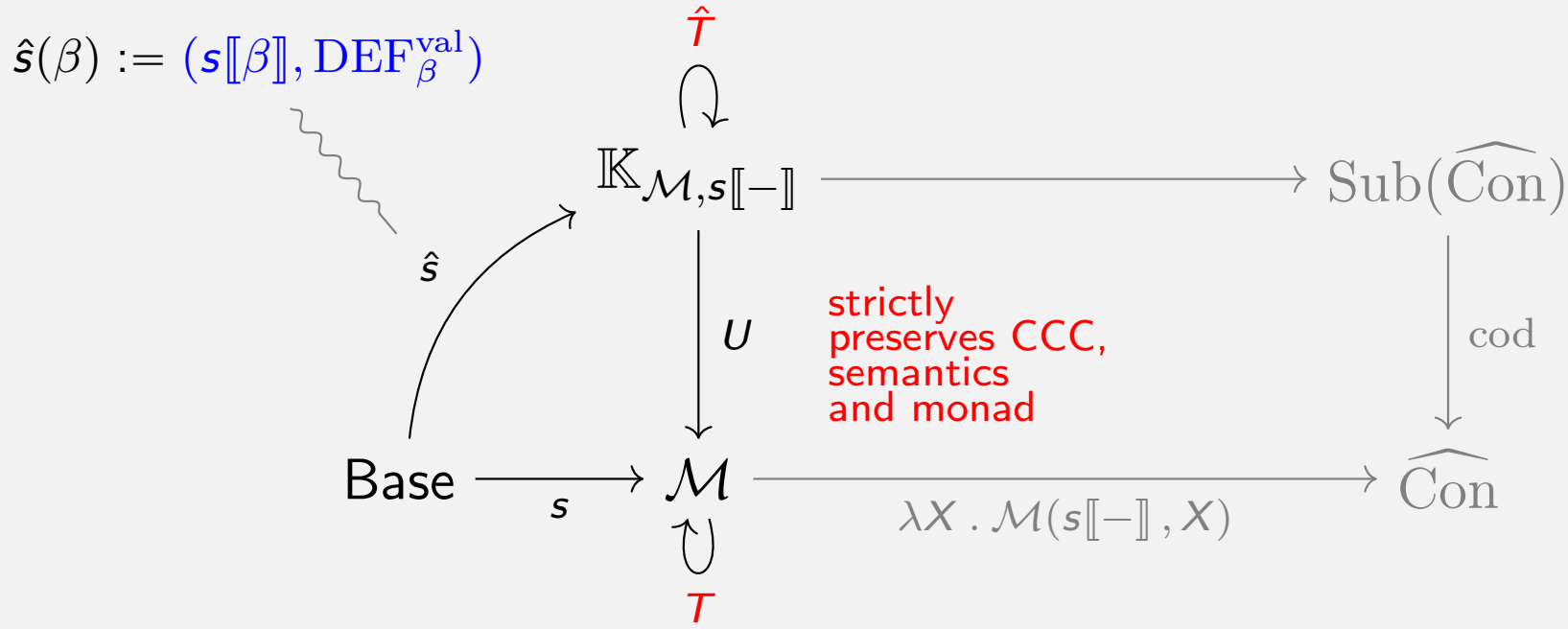


choosing $\hat{T}$ by $\top\top$-lifting



choosing $\hat{T}$ by $\top\top$-lifting ... for every type σ:

$$\hat{s}[\![\sigma]\!] = (s[\![\sigma]\!], \mathrm{DEF}^{\mathrm{val}}_{\sigma}) \quad \text{and} \quad \hat{T}\hat{s}[\![\sigma]\!] = (Ts[\![\sigma]\!], \mathrm{DEF}_{\sigma})$$



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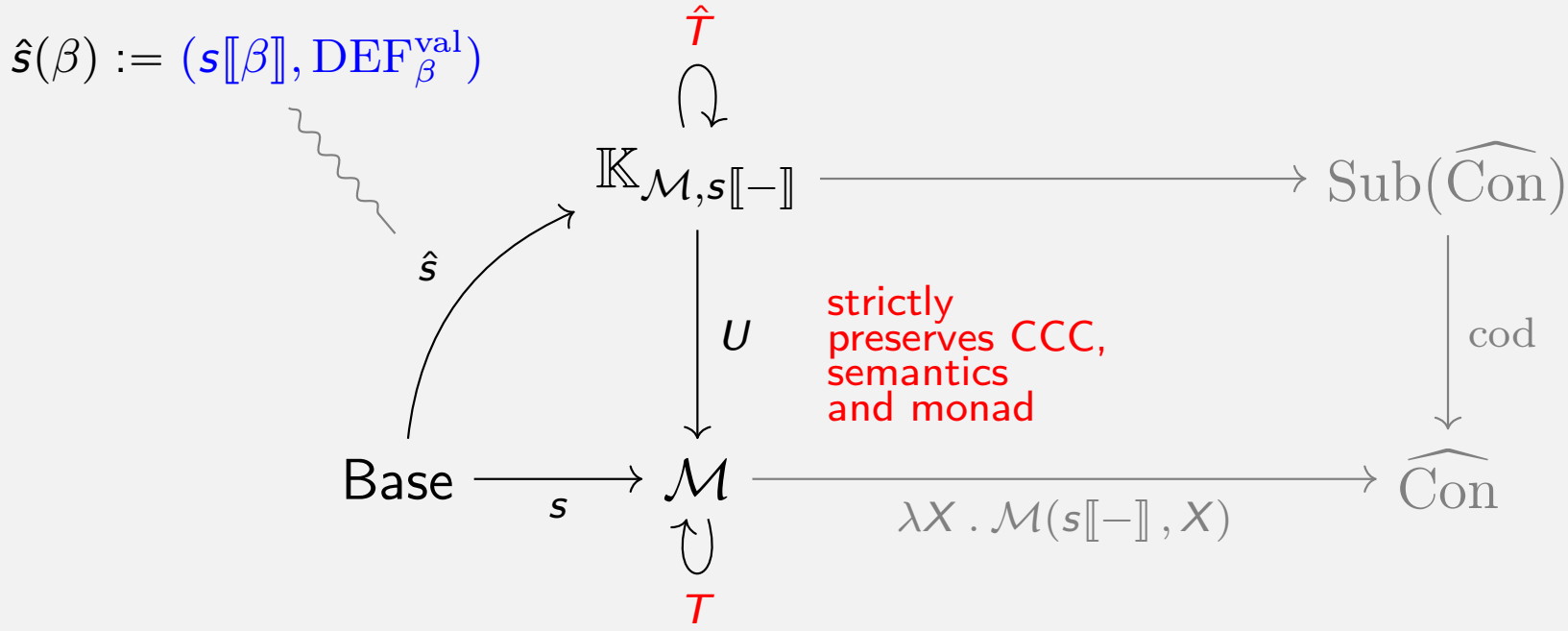

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The route so far

- ✓ the model we've just seen, abstractly
- ✓ restricting to definable morphisms
- 3. the abstract OHR construction $\rightsquigarrow$ follows pattern just seen
- 4. getting full abstraction

The abstract OHR construction

A recipe for removing bad morphisms κ

1. Use relations $\rightsquigarrow$ stops them being morphisms
2. Use concreteness $\rightsquigarrow$ cuts them out function spaces

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$$\begin{array}{c} \mathcal{M} \\ \circlearrowleft \\ T \end{array}$$

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$$\prod_{i \in \mathbb{I}} \begin{pmatrix} \text{predicates} \\ \text{on } \mathbb{B}_i \end{pmatrix}$$

$$\Big\downarrow \text{ fibration for logical relations}$$

$$\prod_{i \in \mathbb{I}} \mathbb{B}_i$$

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NB: $\mathbb{I}$-many categories on RHS $\rightsquigarrow$ $\mathbb{I}$-many relations

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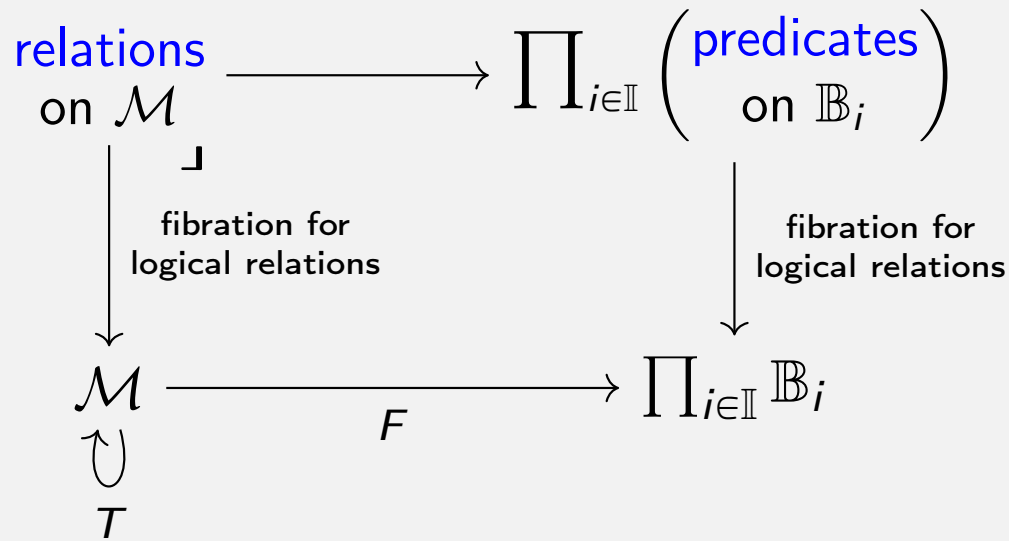

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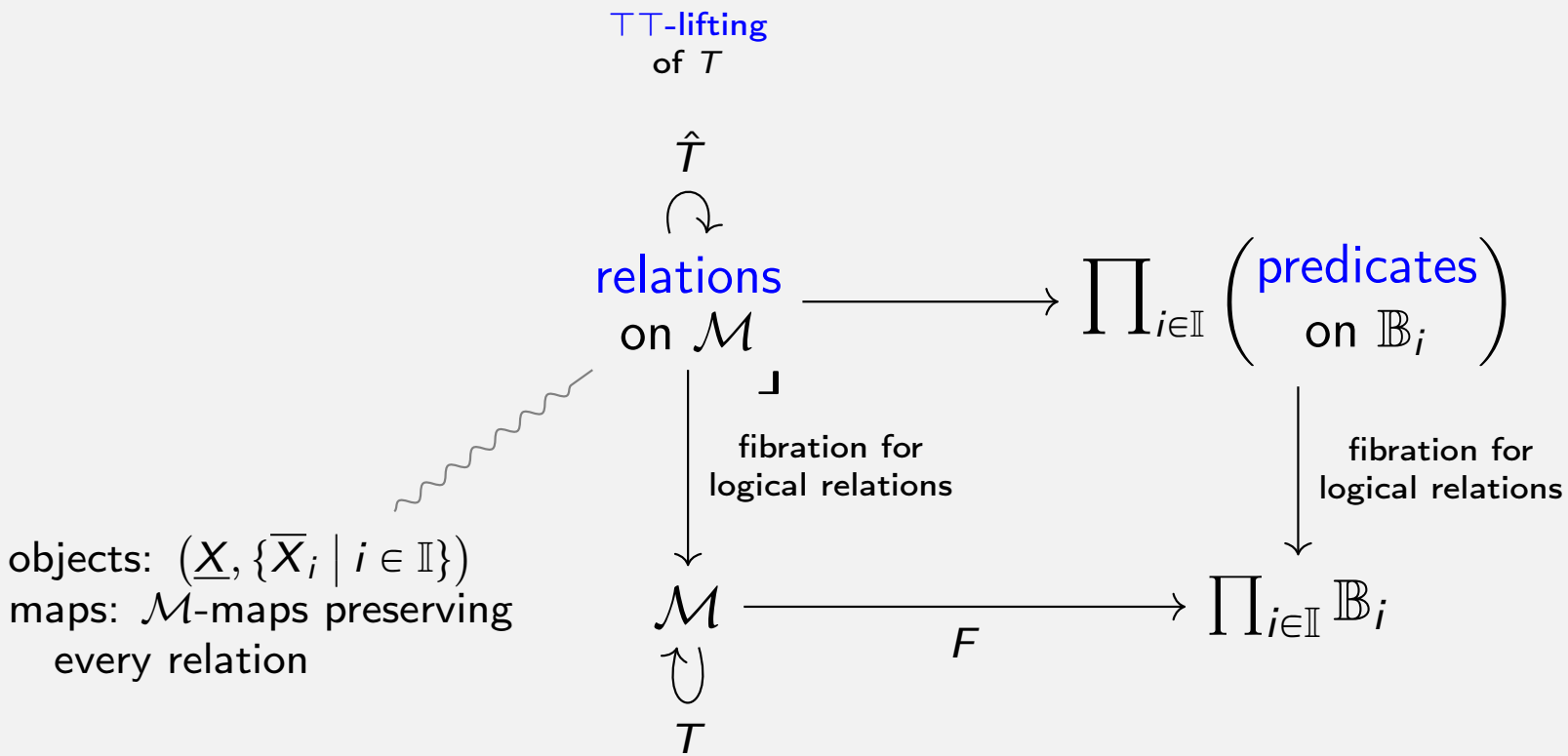

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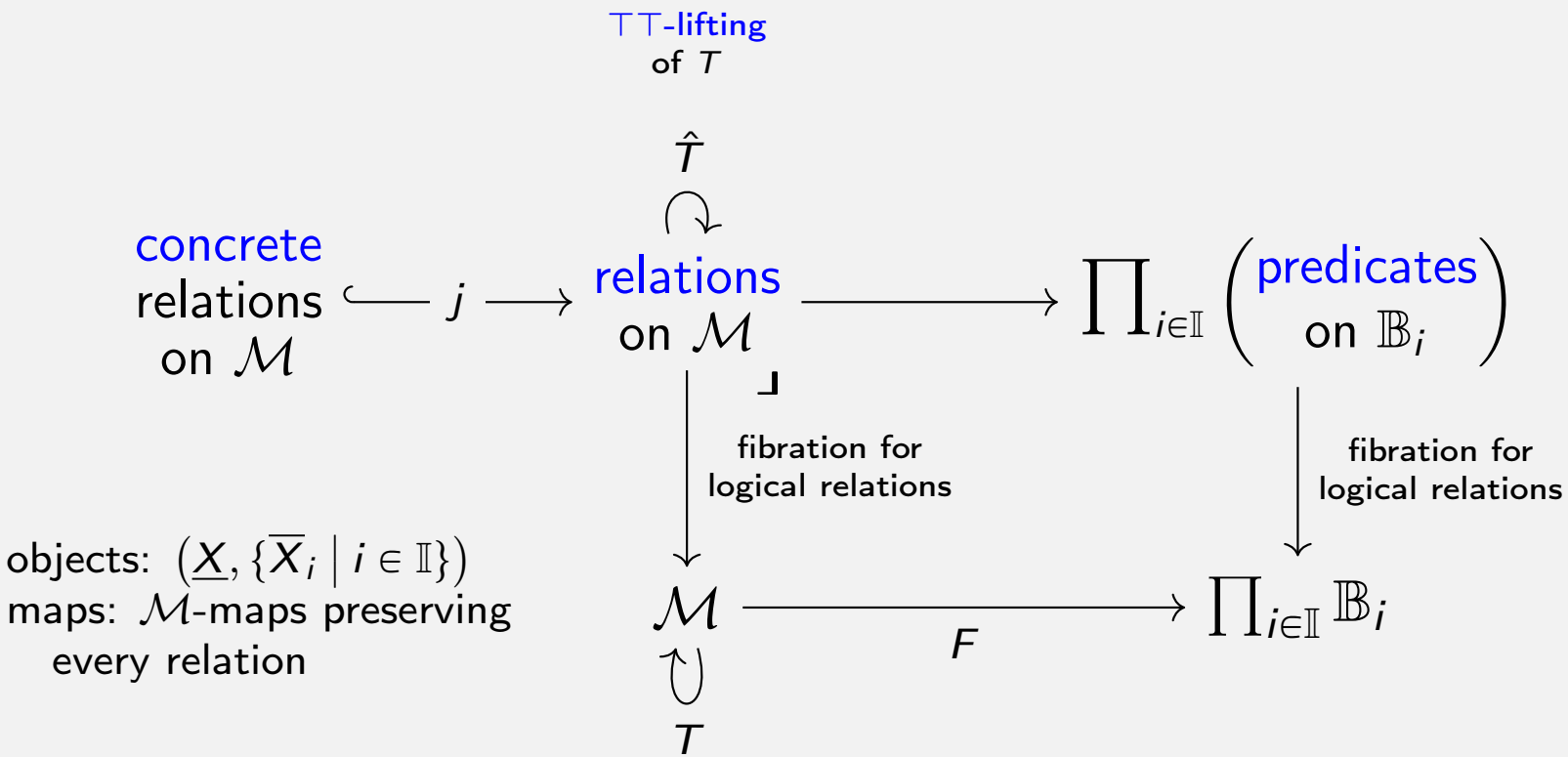

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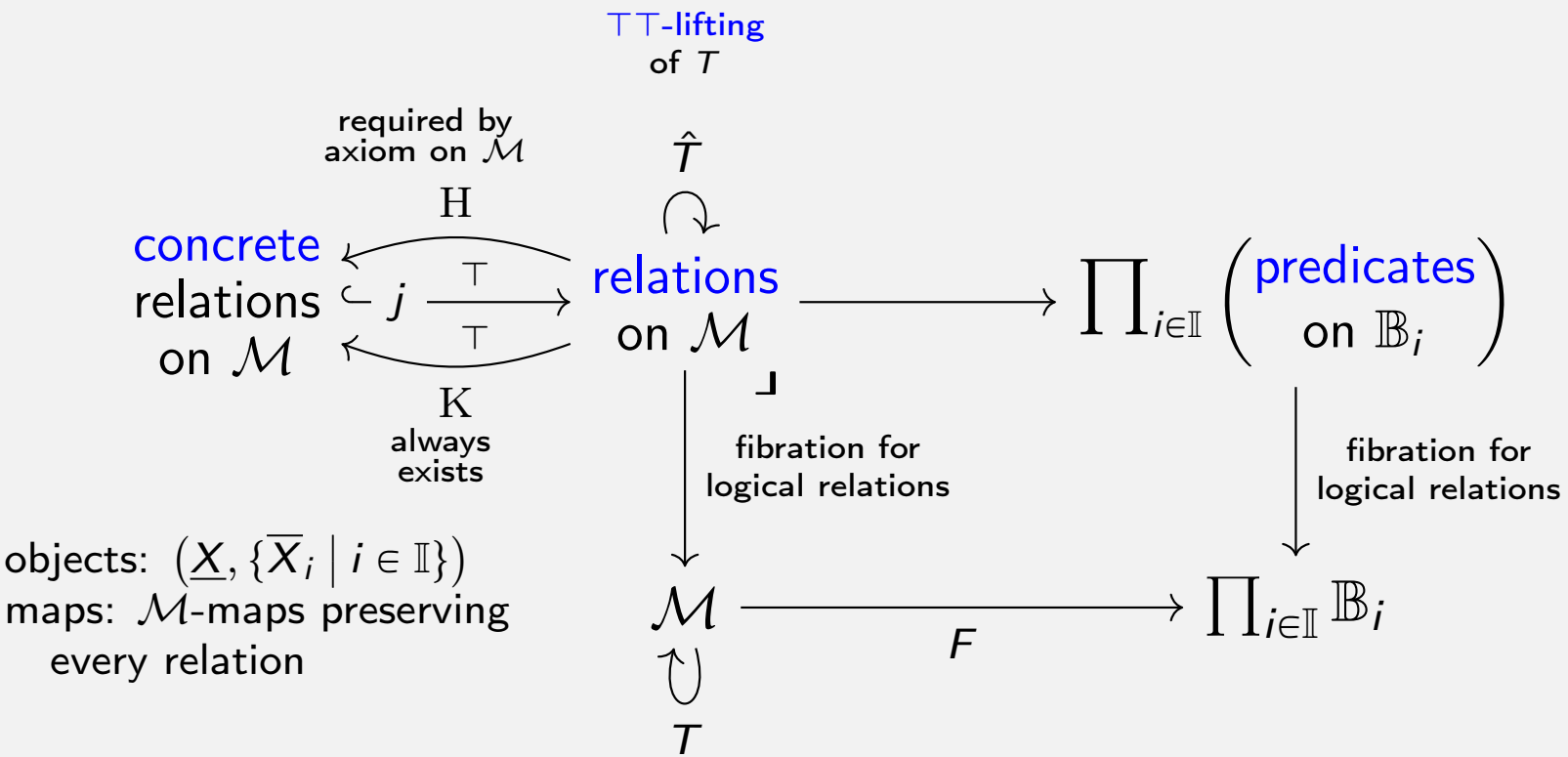

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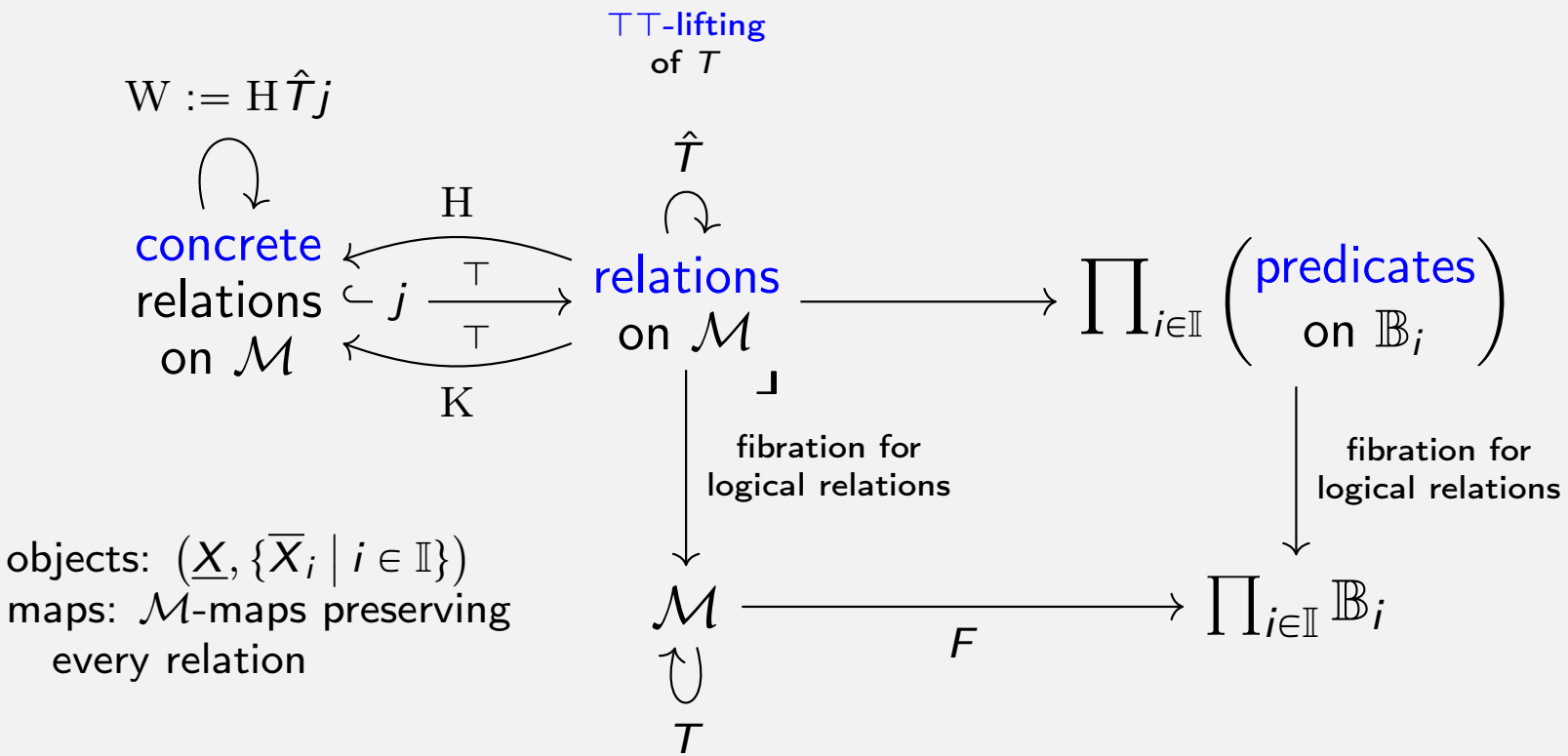

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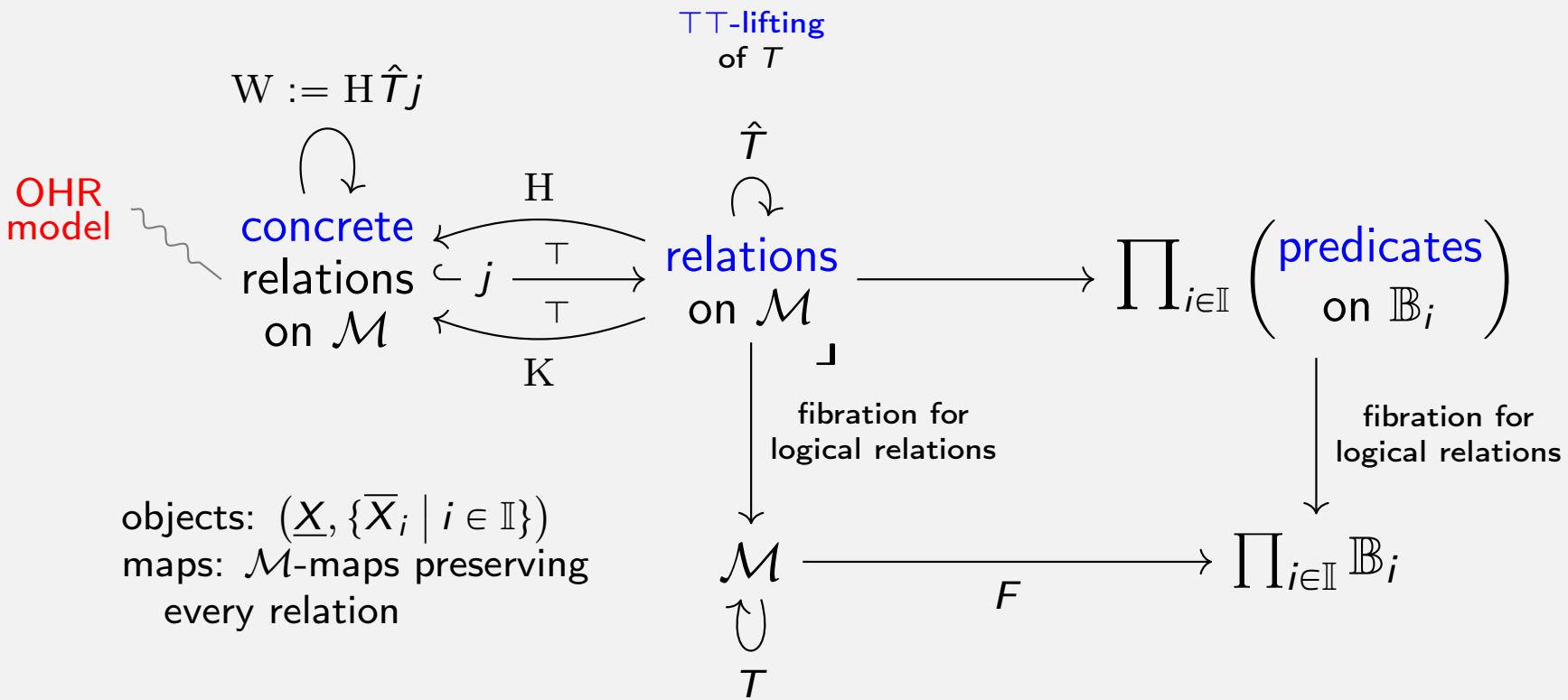

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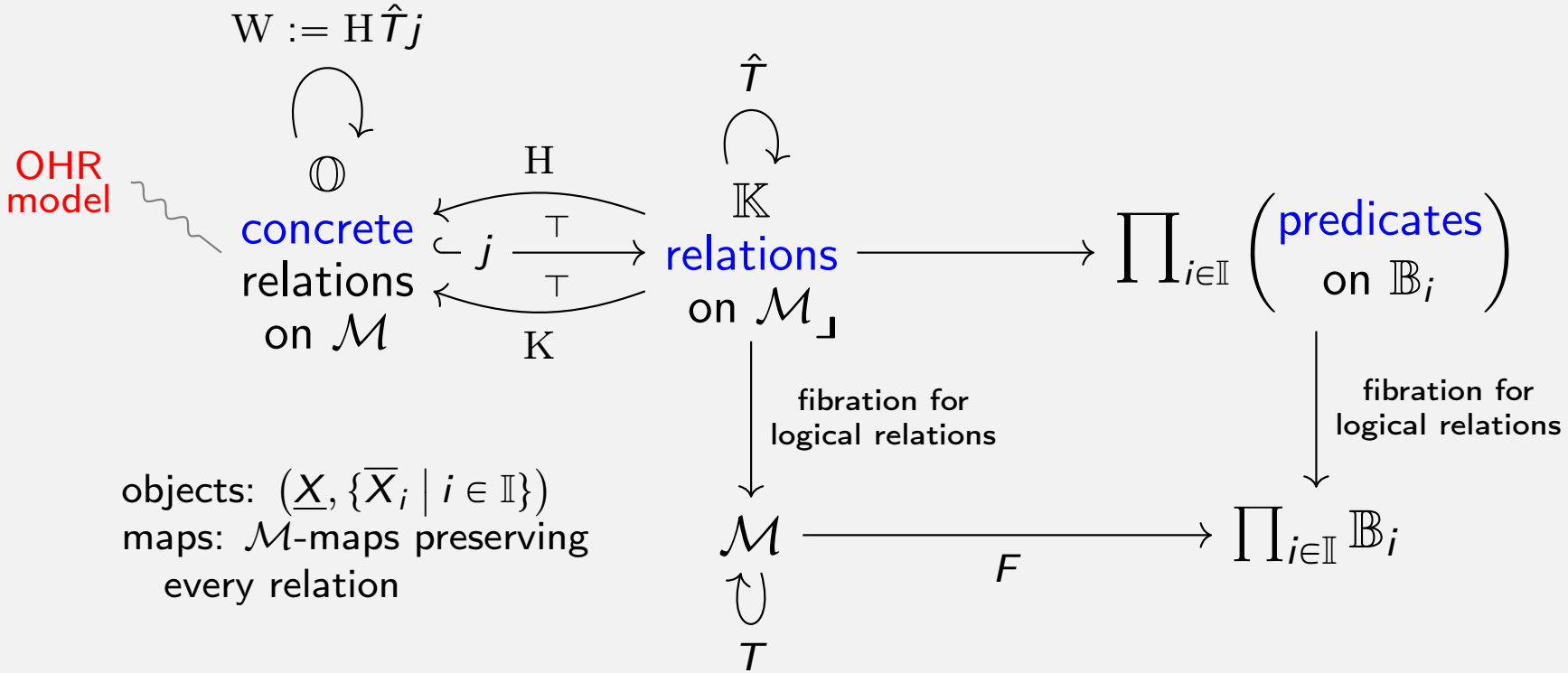

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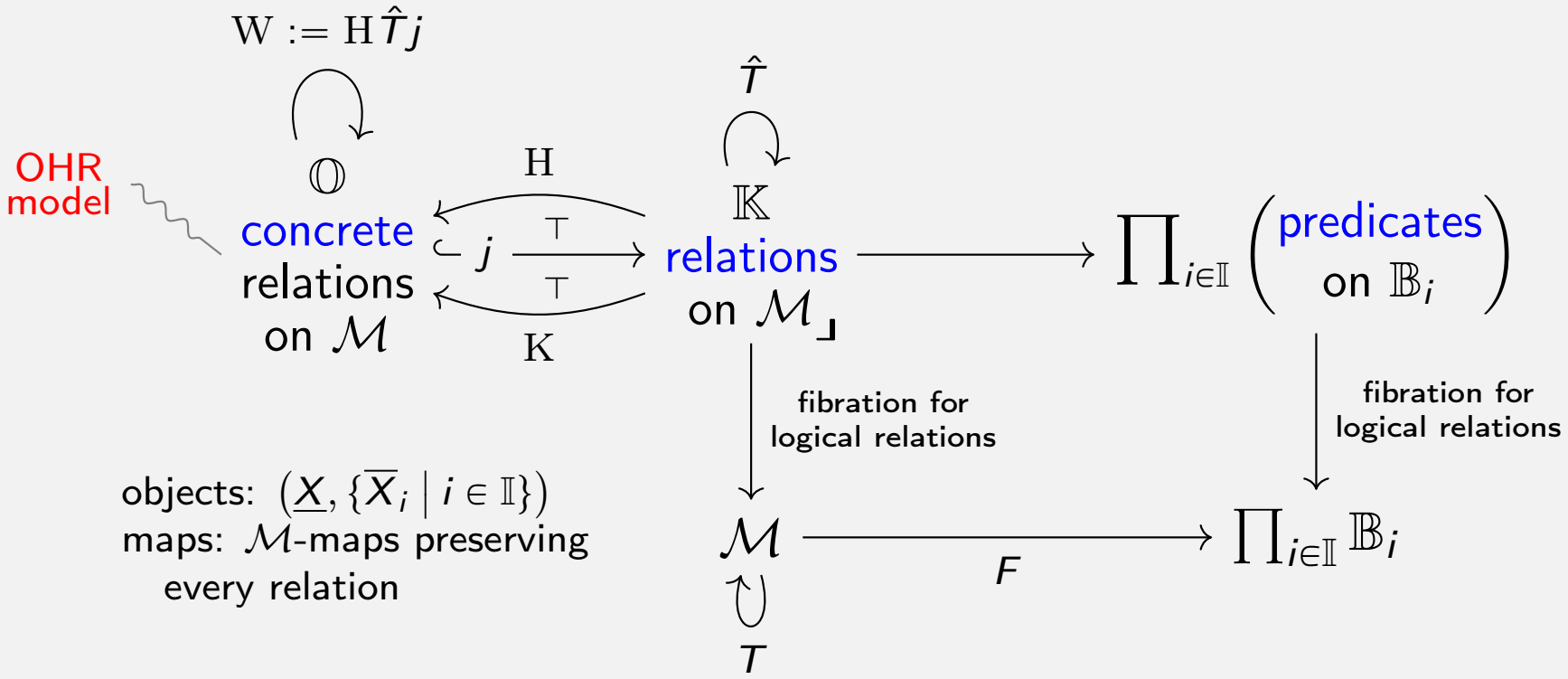

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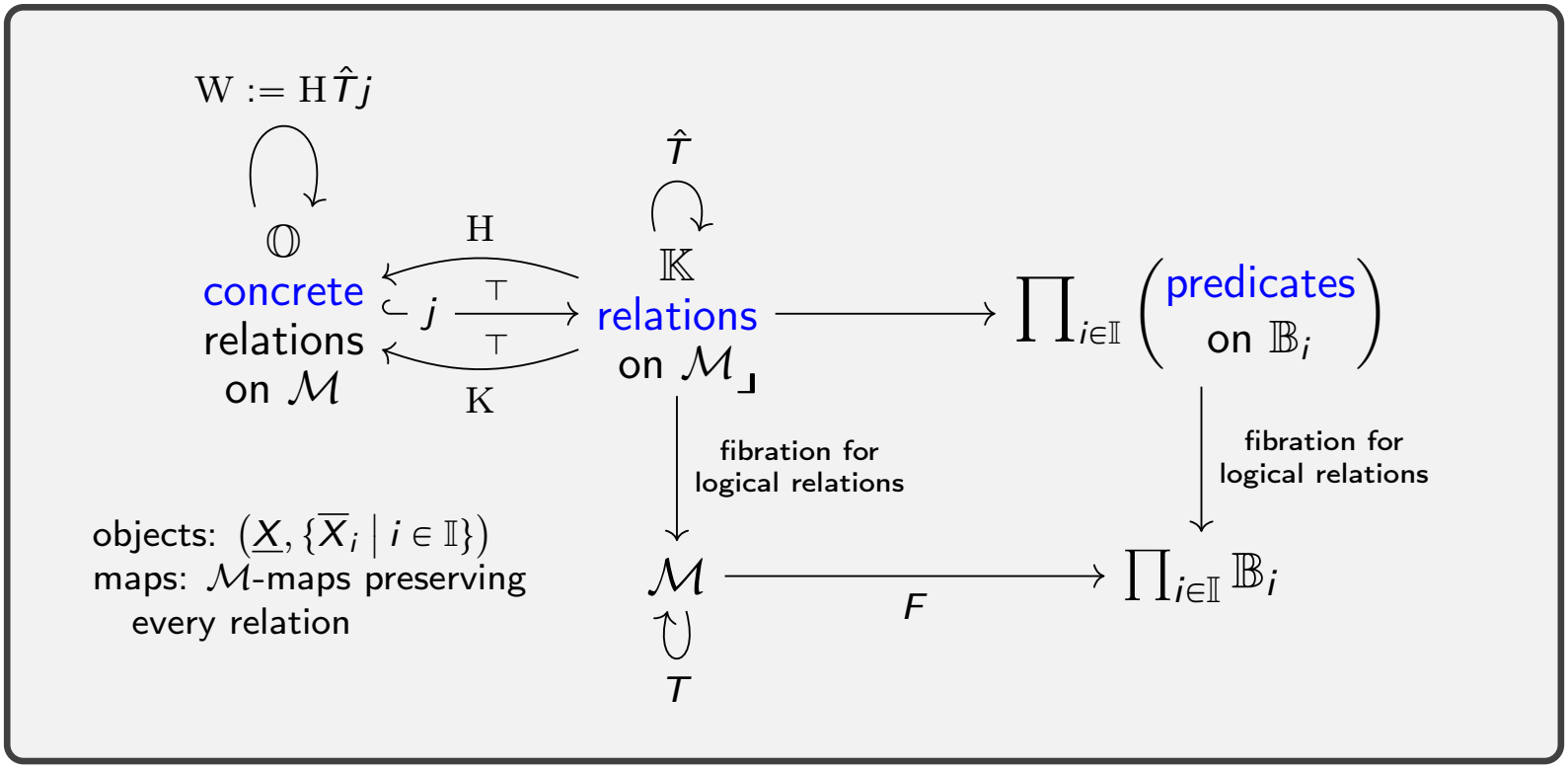

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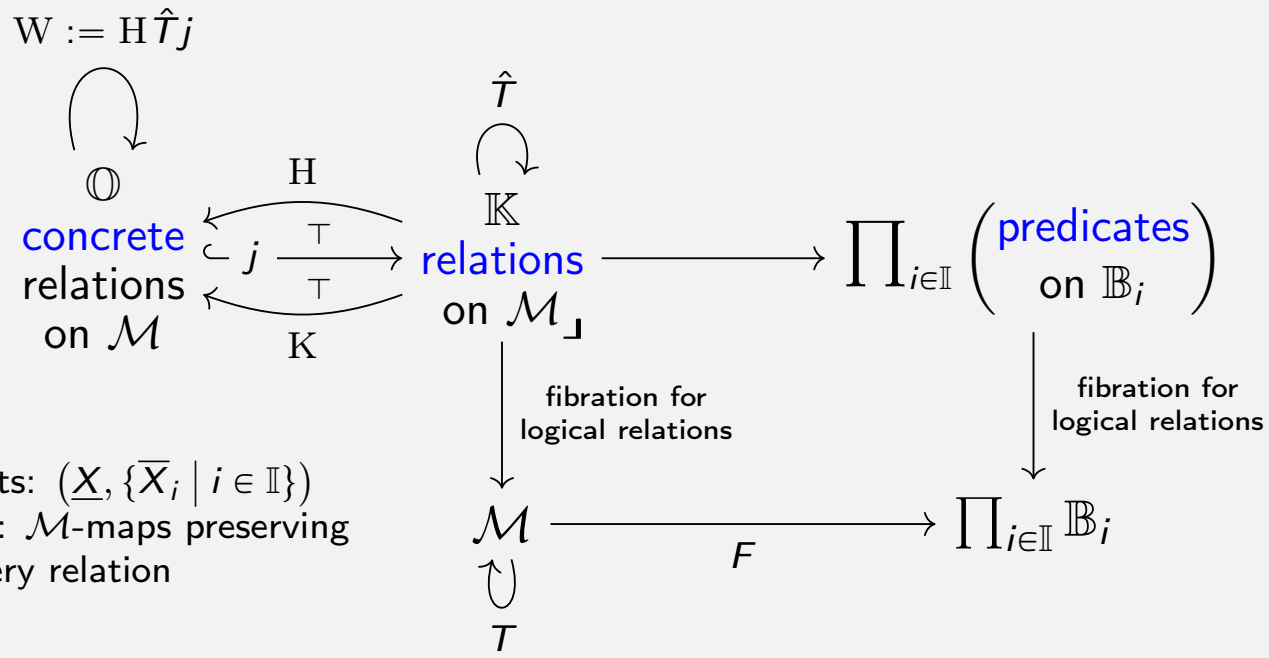

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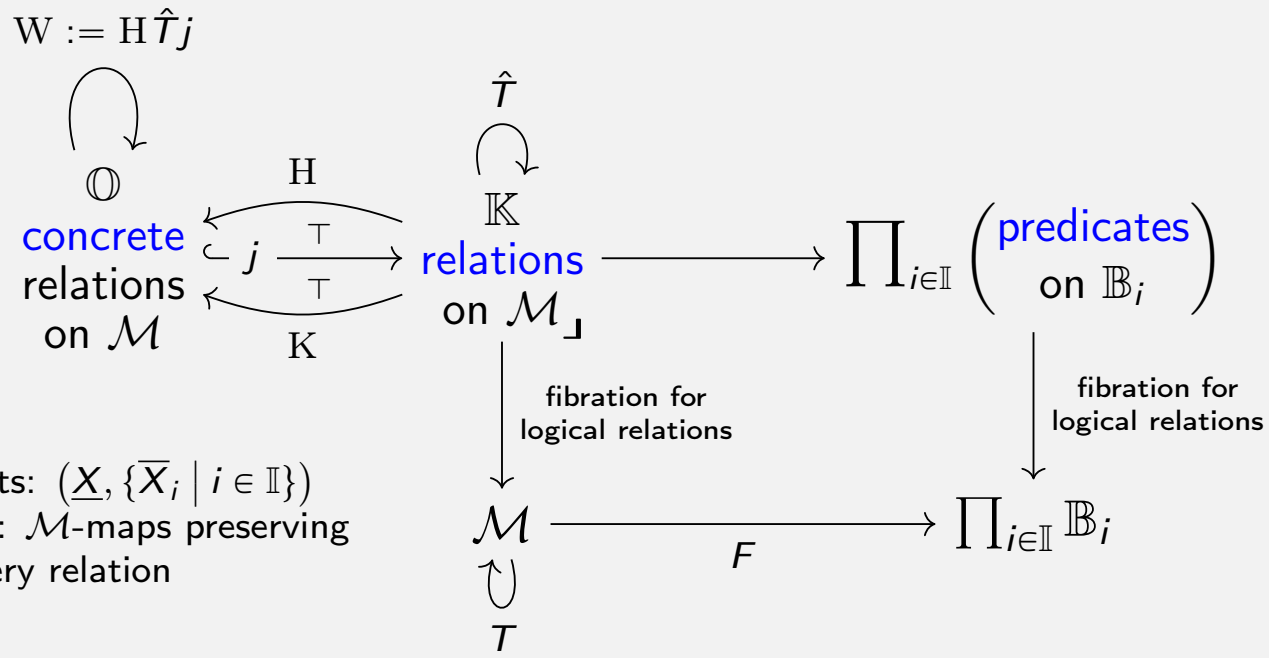

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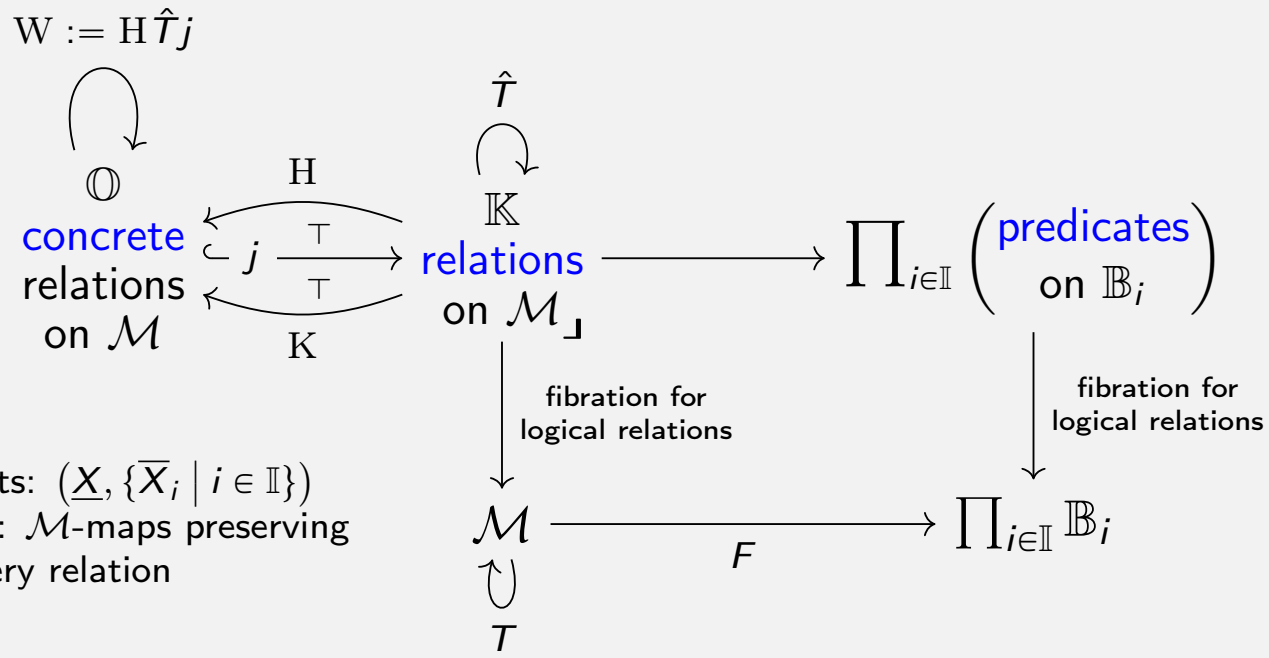

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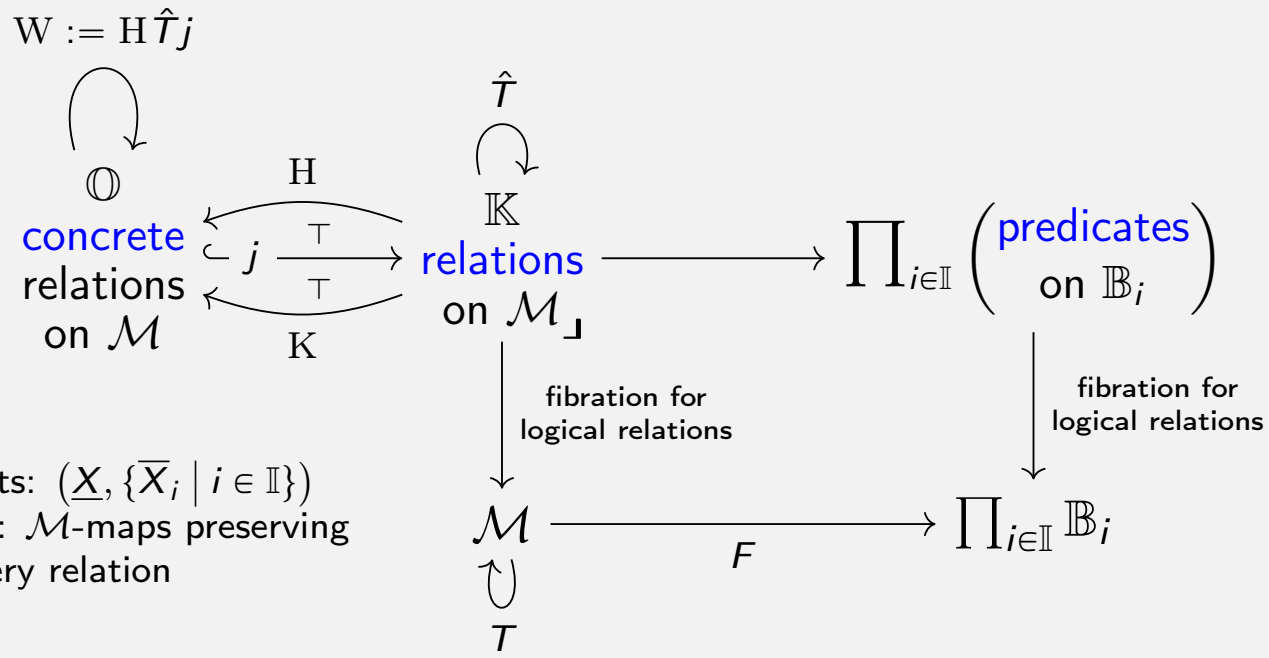

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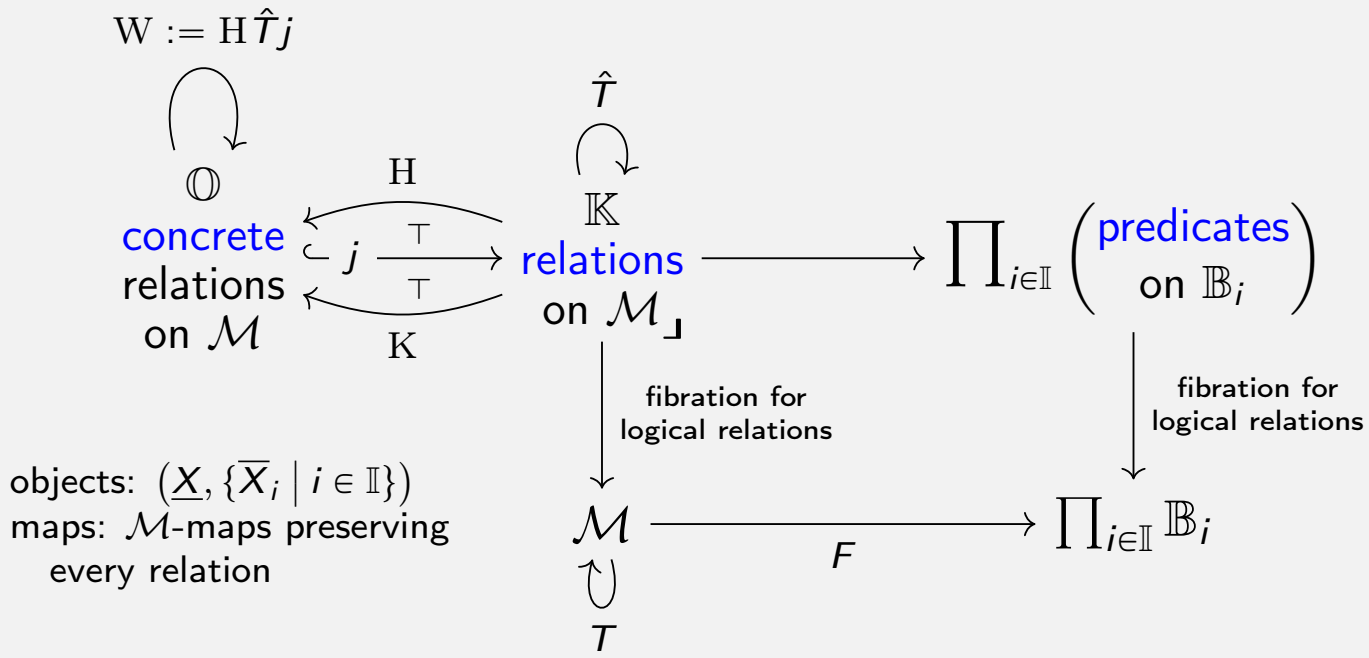

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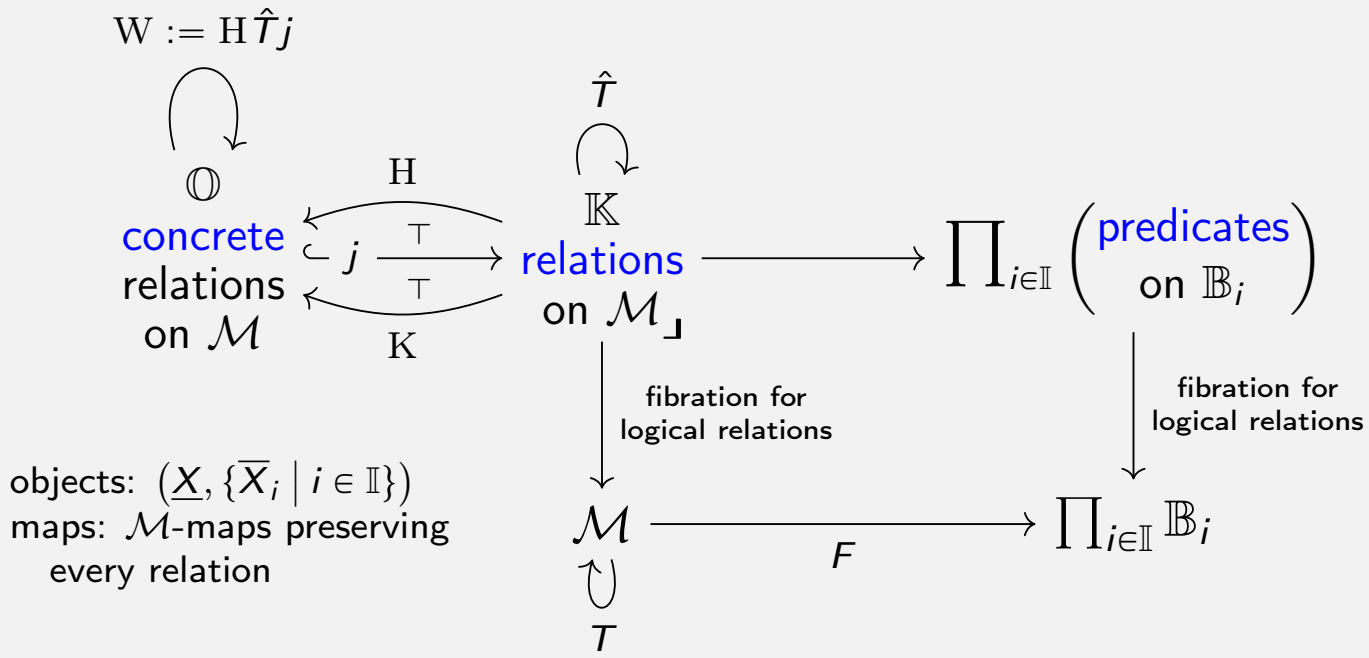

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Strategy: see what we need as we go along!

Getting full abstraction

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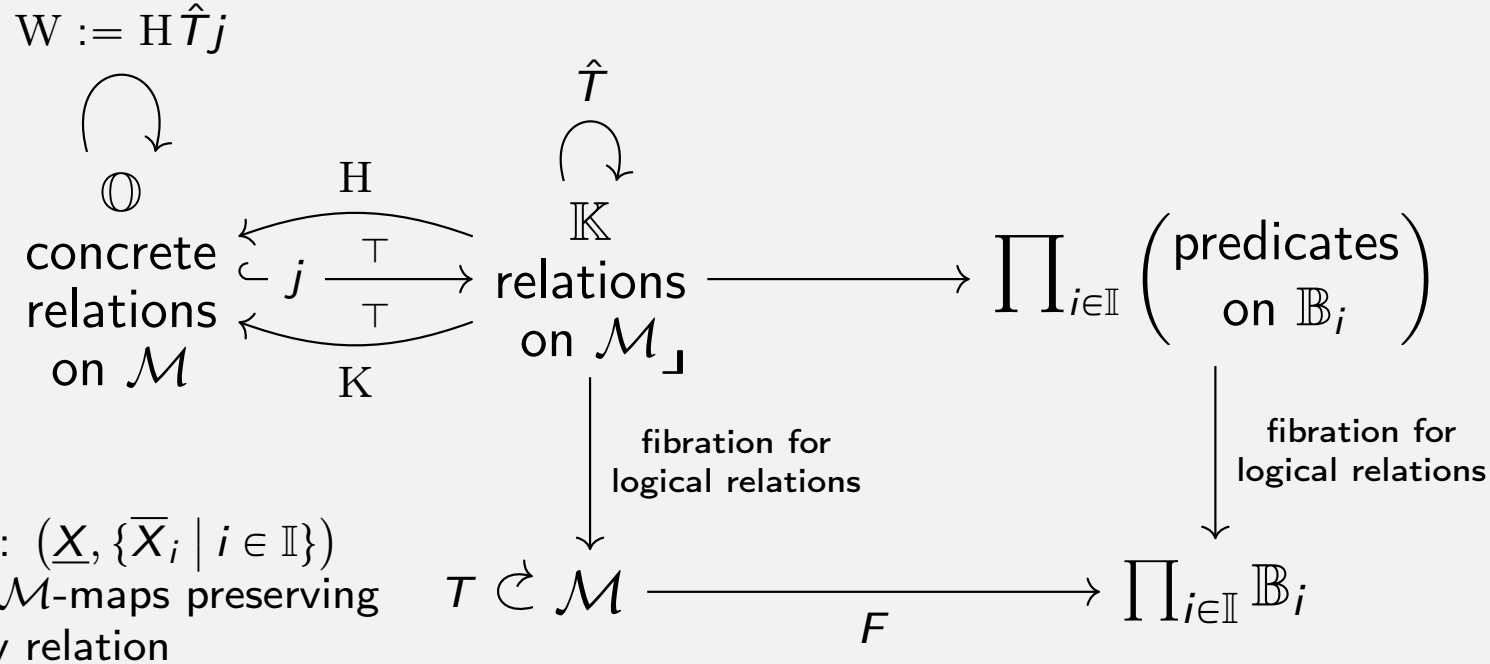

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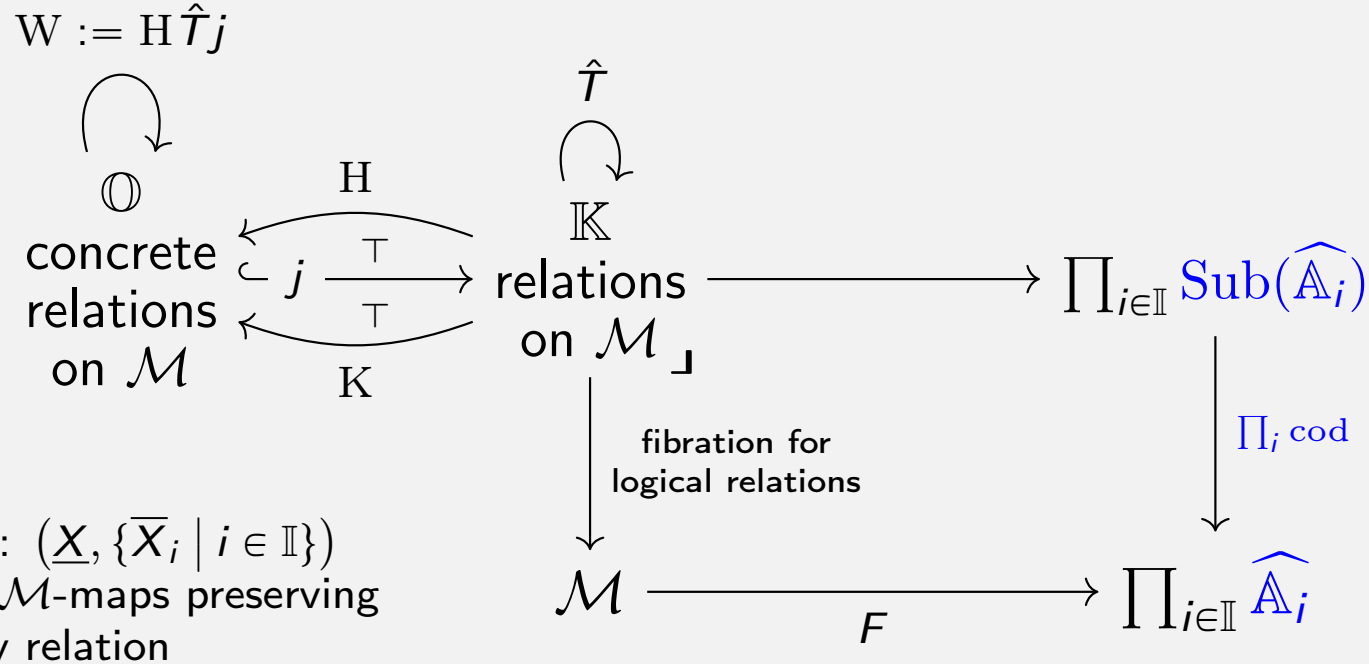

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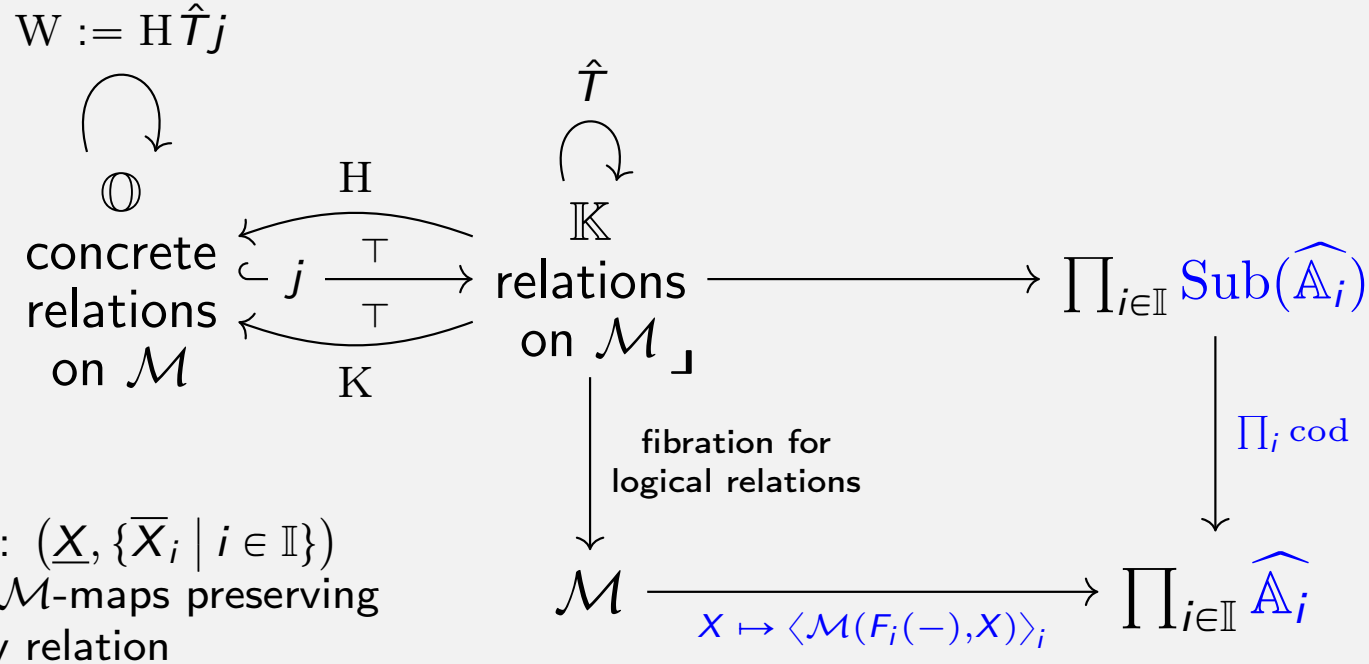

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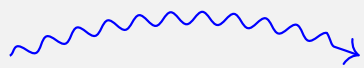
define a model $(\mathbb{O}, \mathrm{W}, \hat{s})$  construct the
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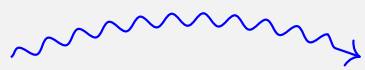

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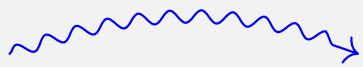

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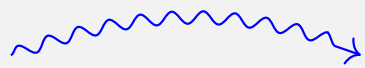

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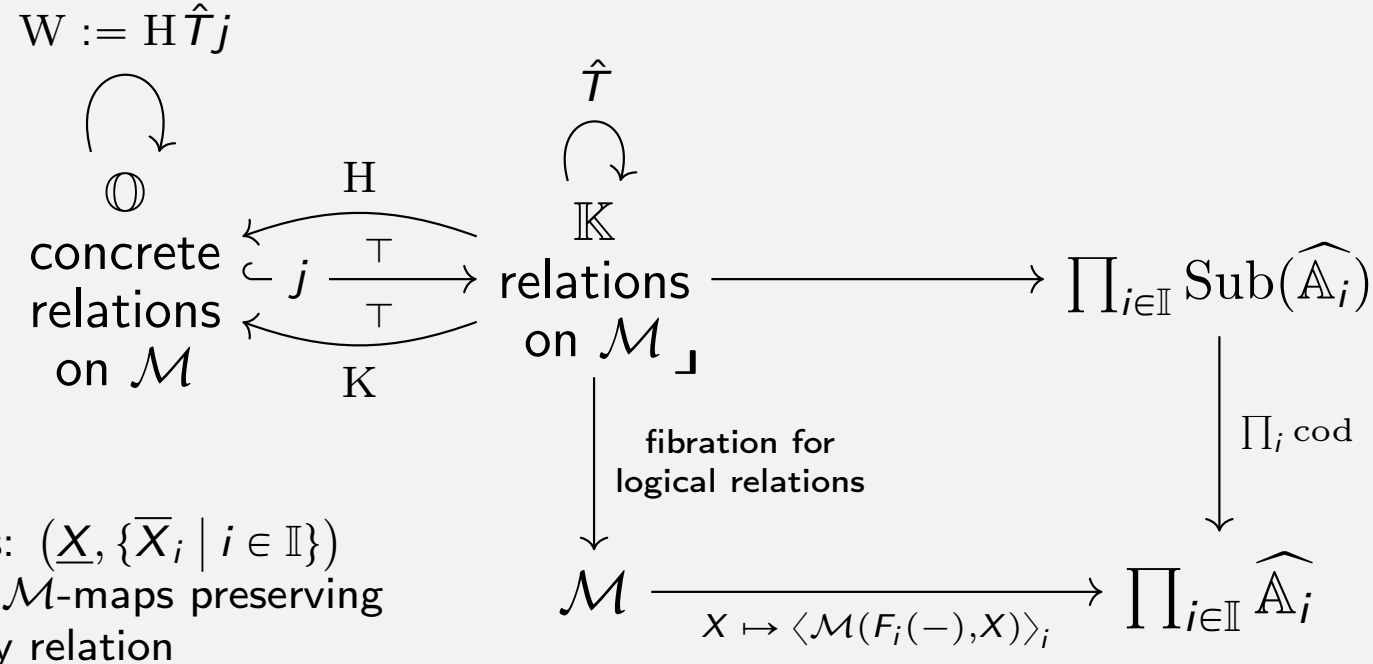
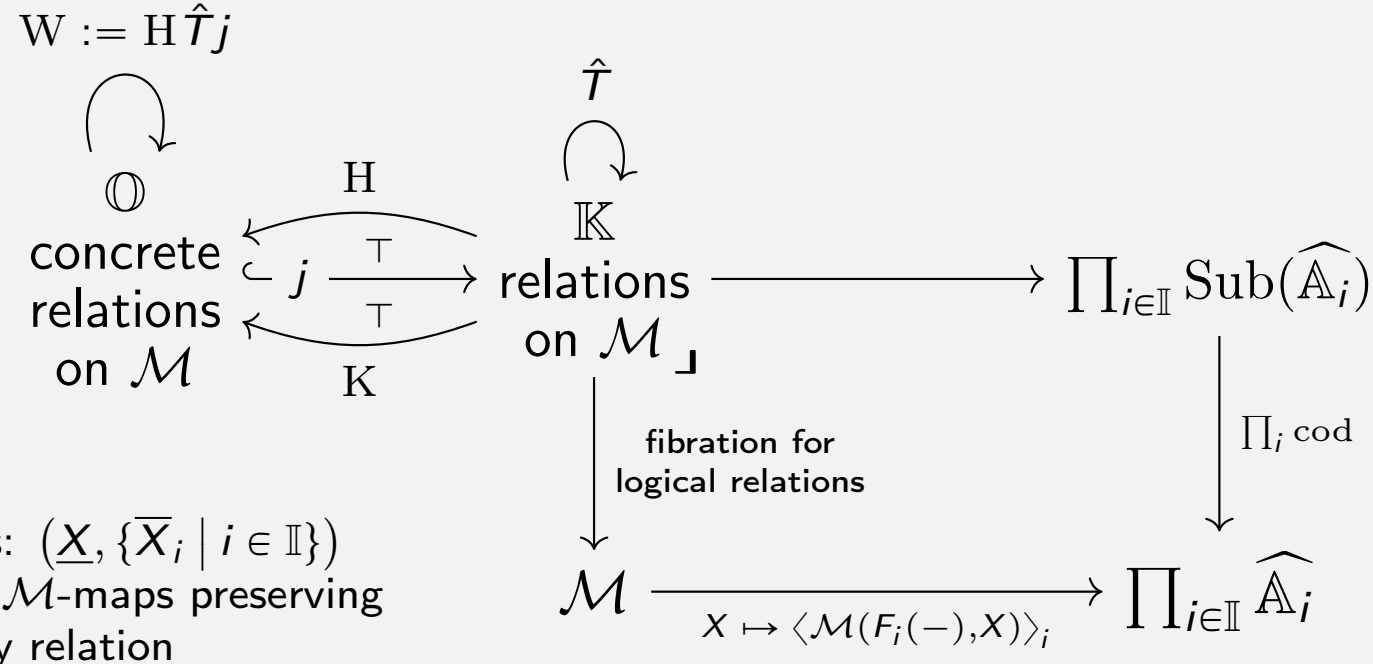
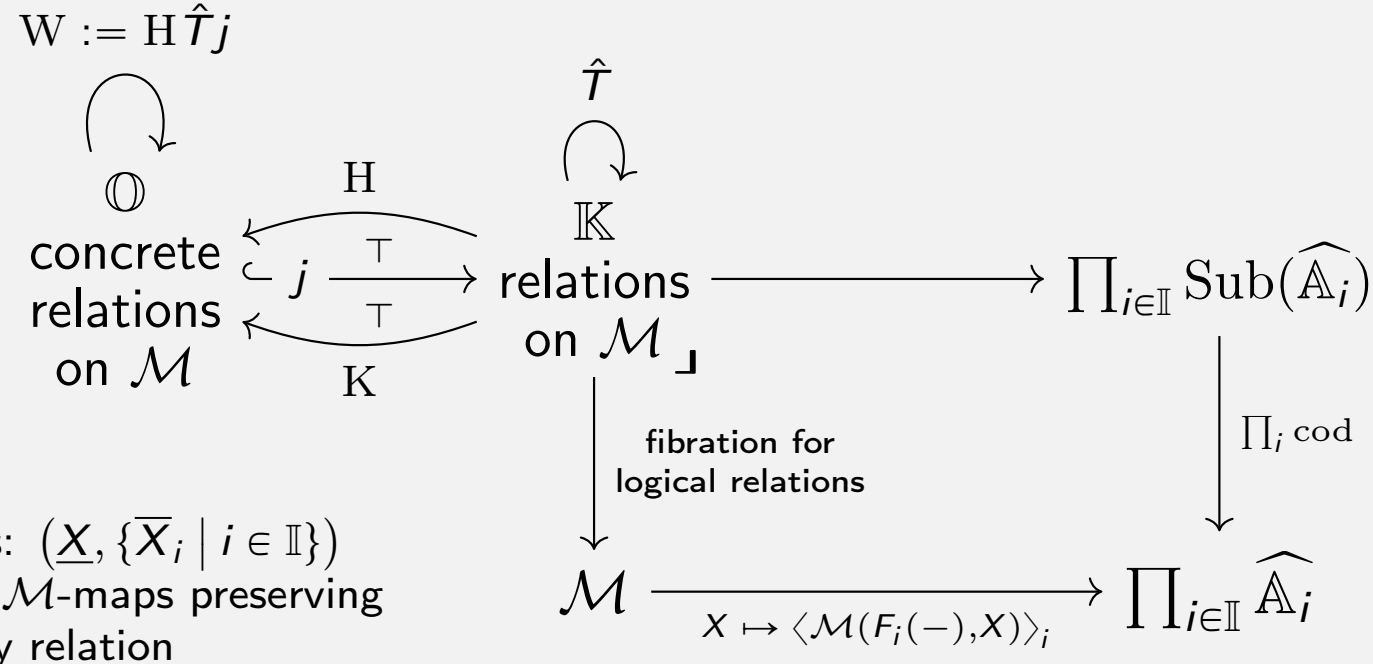
Solution: relations on $\mathbb{O}$ are relations on $\mathcal{M}$!



objects: $\left(\underline{X}, \{\overline{X}_i \mid i \in \mathbb{I}\}\right)$
maps: $\mathcal{M}$-maps preserving every relation



Have $U : \mathbb{O} \to \mathcal{M}$



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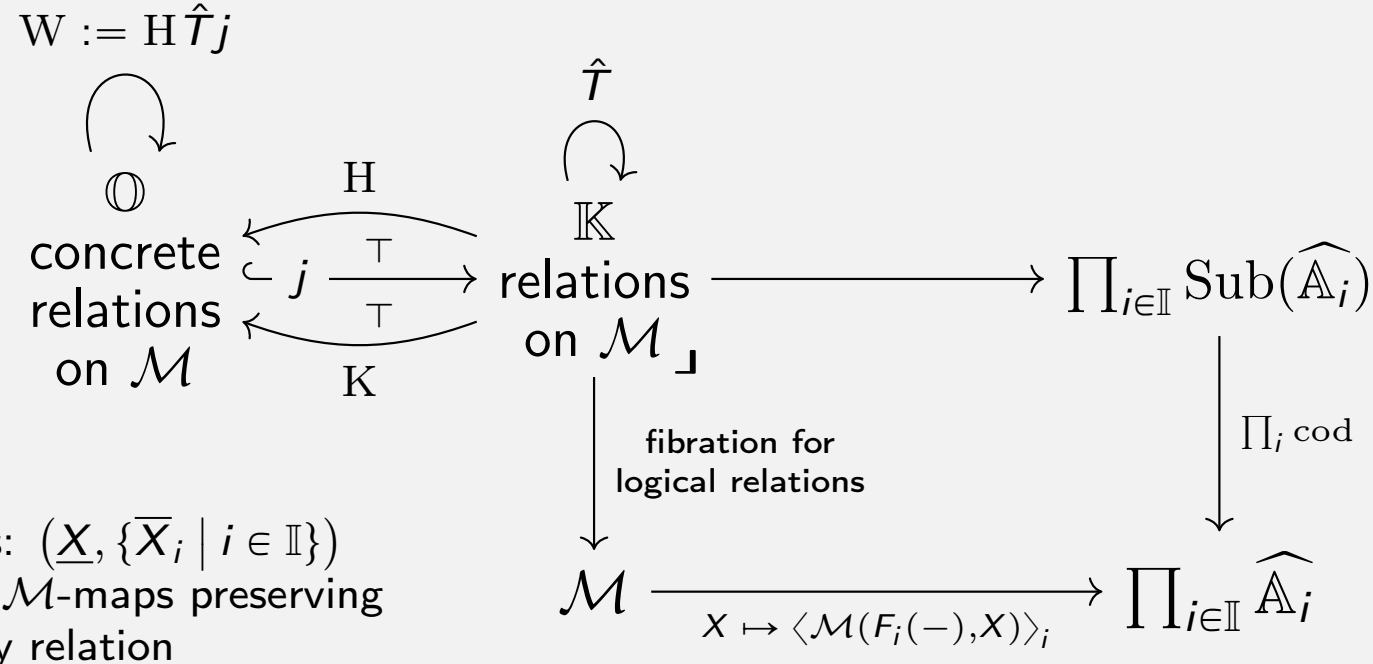

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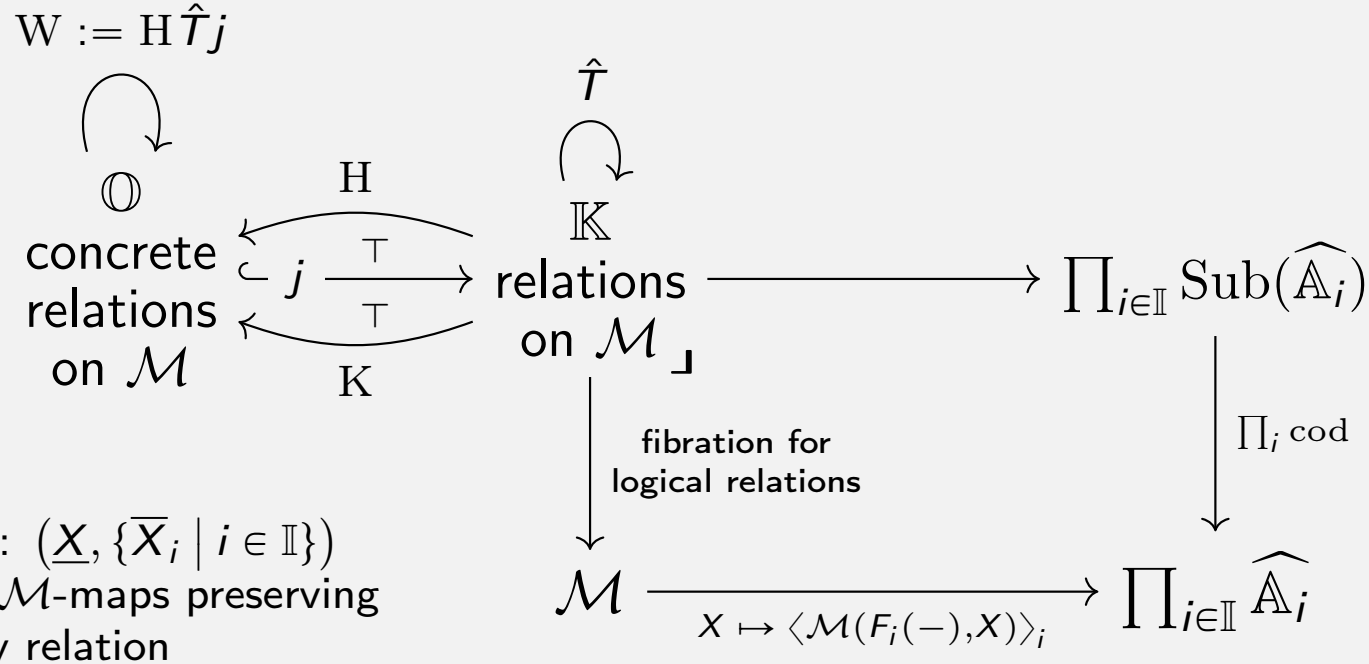

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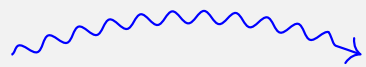

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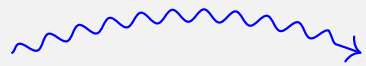

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Then: will get $\exists i_0 \in \mathbb{I}$

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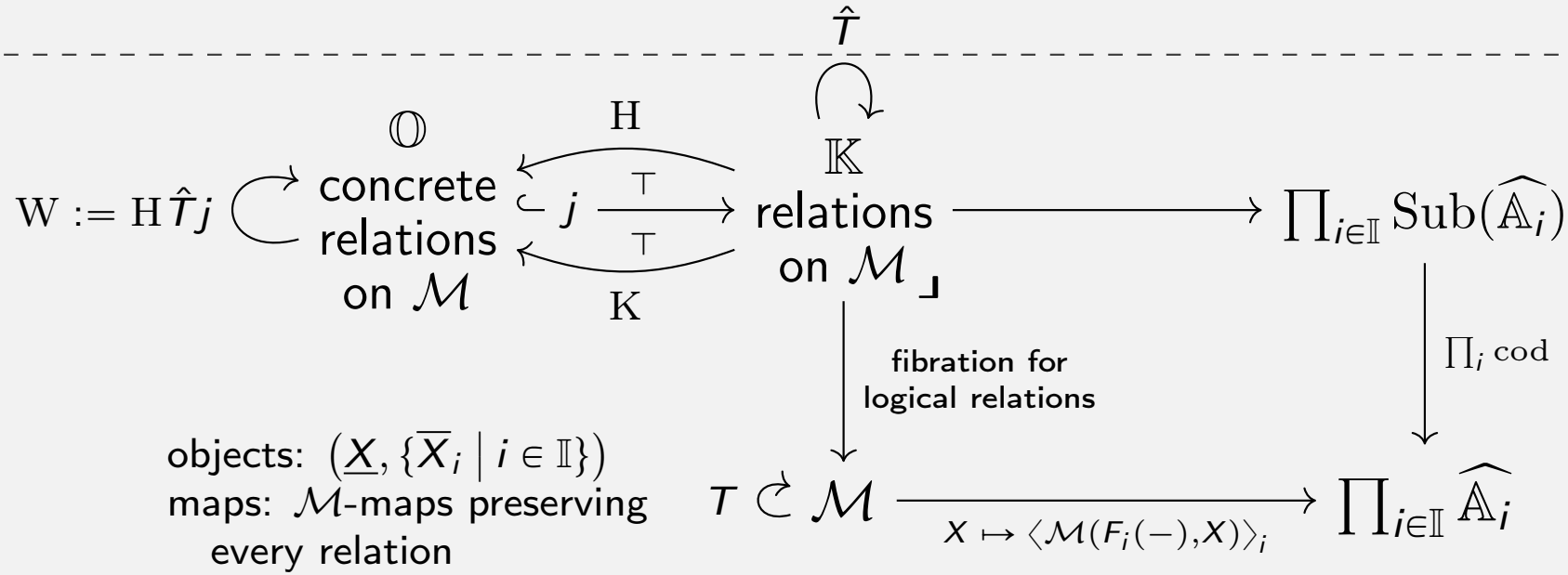

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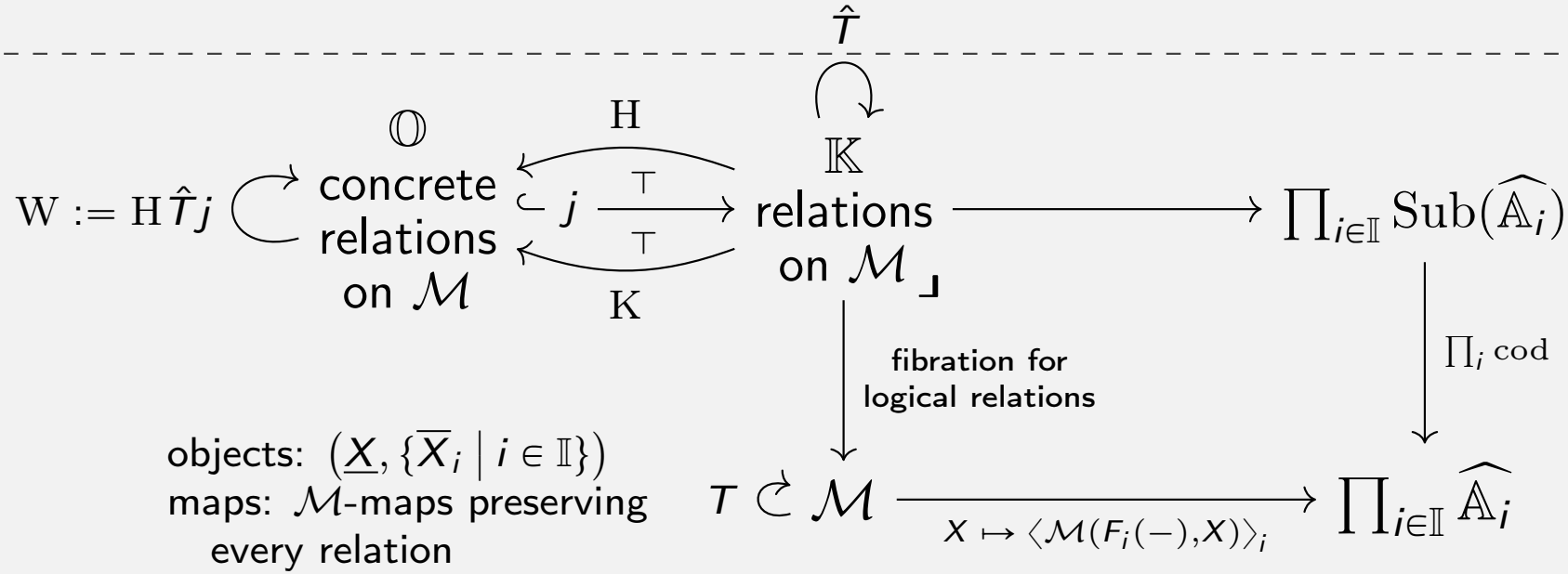

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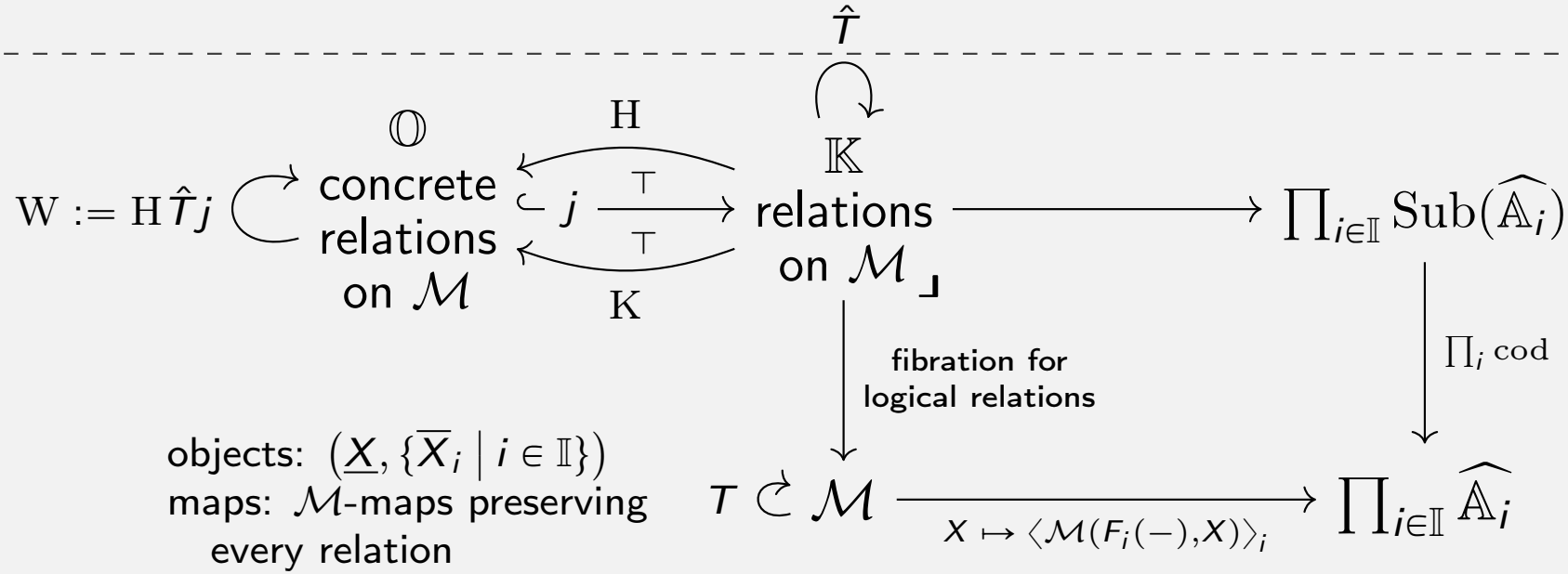

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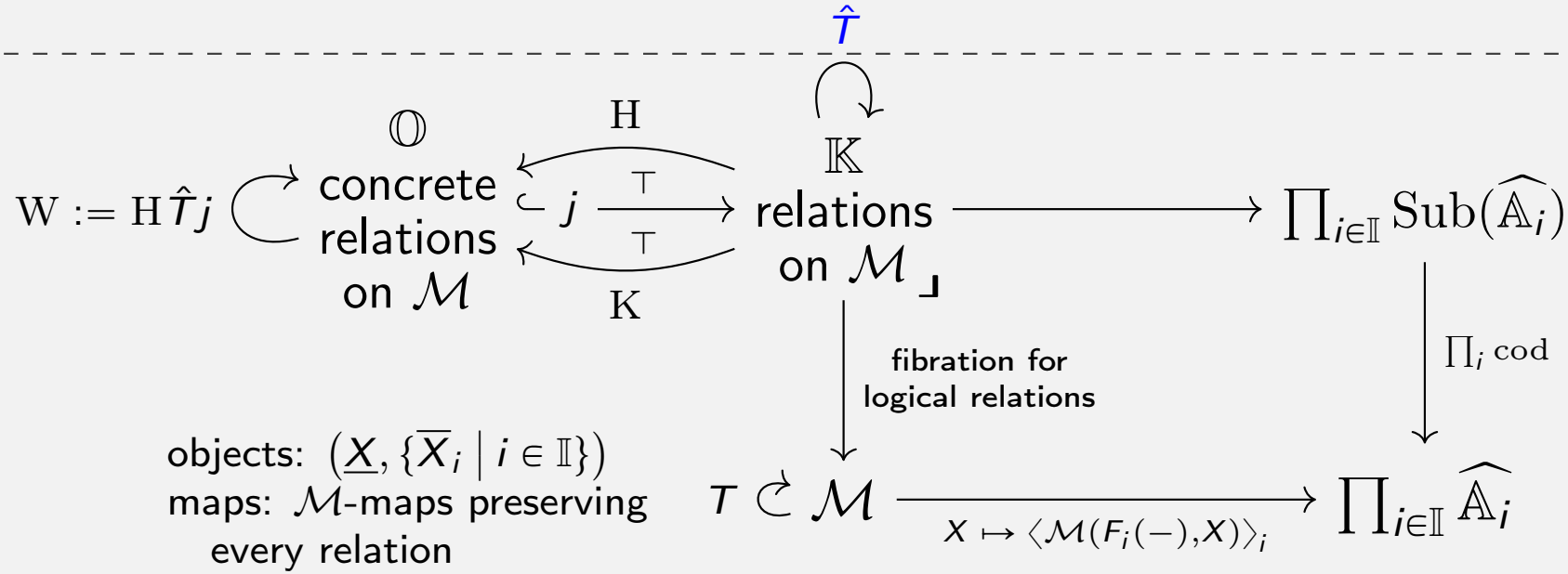

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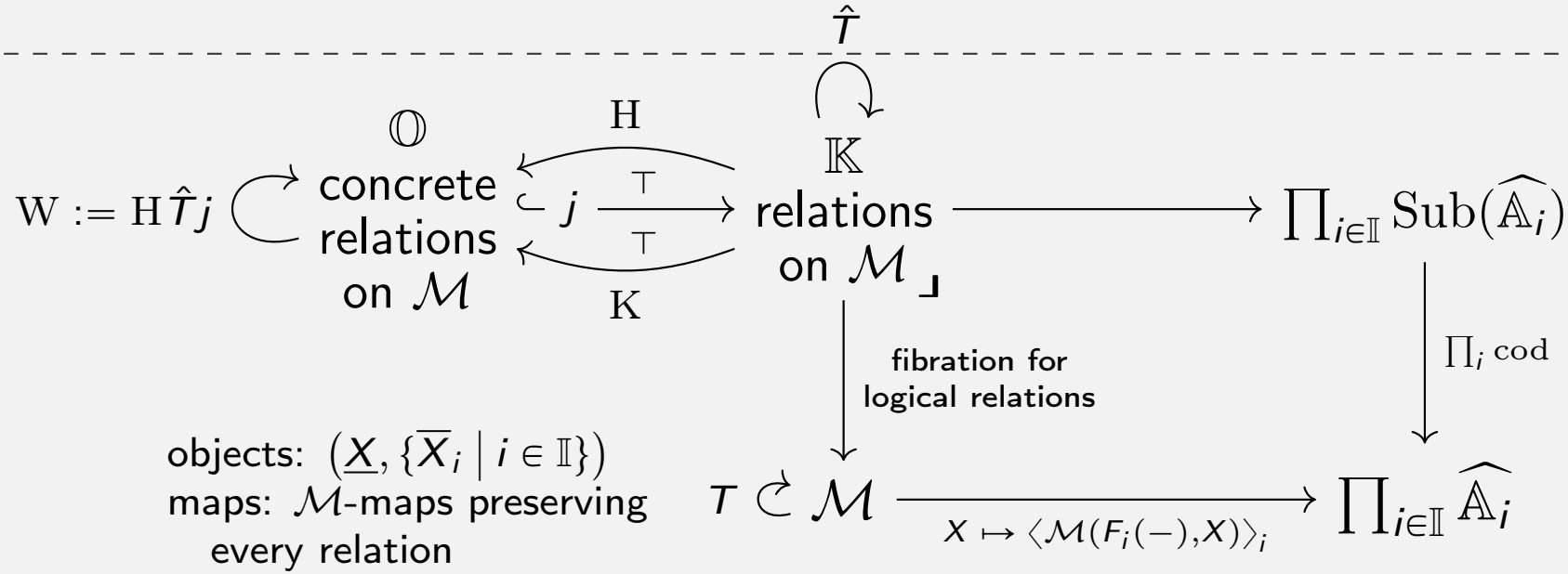

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   $\rightsquigarrow$ to define our semantic interpretation in $\mathbb{O}$

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preprint at cs.ox.ac.uk/people/philip.saville/home.html