A semantic approach to coherence for cartesian closed bicategories

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This talk:

- based on work from my PhD (and a bit after)
- about: using ideas from programming languages / categorical logic to prove coherence theorems... and hence simplify calculating in Esp
- aim: convey big picture / main ideas.
  See associated papers / my thesis for more.

The bicategory Esp

cf. models of linear logic

[$!\mathbb{C}$ = free monoidal category on $\mathbb{C}$]

__objects__ : categories $\mathbb{C}, \mathbb{D}, \ldots$

__hom-categories__ : $\mathrm{Esp}[\mathbb{C}, \mathbb{D}] := \mathrm{Cat}(!\mathbb{C}, \widehat{\mathbb{D}})$

$$\mathrm{Esp}(\mathbb{1}, \mathbb{1}) = [\mathrm{Bij}, \mathrm{Set}]$$

__composition__ : some coend

$$\begin{tikzcd}
 & \mathbb{D} \arrow[dr, "Q", "\shortmid"{marking}] & \\
\mathbb{C} \arrow[ur, "P", "\shortmid"{marking}] \arrow[rr, "Q \circ P"', "\shortmid"{marking}] & & \mathbb{E}
\end{tikzcd}$$

CHALLENGE: calculations are hard!

What would we like to do?

<u>lemma</u>: in a cartesian closed category $X \Rightarrow X$ has a monoid structure.

Pf: big diagram chase. //

reduce the difficulty of calculating

<u>lemma</u>: In any "cartesian closed bicategory" $X \Rightarrow X$ has a pseudomonoid structure.

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times M"] \arrow[dr, bend right, "\simeq"'] & M \times M \arrow[d, "m"] & M \times 1 \arrow[l, "M \times e"'] \arrow[dl, bend left, "\simeq"] \\
& M &
\end{tikzcd}
\qquad \lambda: \cong, \quad \rho: \cong$$

$$\begin{tikzcd}
(M \times M) \times M \arrow[r, "\simeq"] \arrow[d, "m \times M"'] & M \times (M \times M) \arrow[r, "M \times m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

+ coherence axioms

How do we do it?

Two observations:

1. Esp has cartesian closed structure (FGHW)
2. cartesian closed categories have an "internal language" + this is normalising.

get..., a form of coherence

OBSERVATION 1: cc-structure of Esp.

<u>def</u>: a <u>ccc</u> is a category $\mathbb{C}$ equipped with

$$\prod_{i=1}^{n} A_i \quad , \quad A \Rightarrow B$$

for every $A_1 \ldots A_n$, A, B, together with natural isos

$$\mathbb{C}(X, \prod_{i=1}^{n} A_i) \cong \prod_{i=1}^{n} \mathbb{C}(X, A_i)$$

$$\mathbb{C}(X \times A, B) \cong \mathbb{C}(X, A \Rightarrow B)$$ //

eg//

if you have a nice enough comonad ! on a smcc $\mathbb{C}$ then $\mathbb{C}_!$.

Esp has a version of this structure:

$$\bigotimes_{i=1}^{n} \mathbb{C}_i := \sum_{i=1}^{n} \mathbb{C}_i \quad , \quad \mathbb{C} \Rightarrow \mathbb{D} := (!\mathbb{C})^{op} \times \mathbb{D}$$

Then you can calculate:

$$!(\mathbb{C}+\mathbb{B}) \simeq !\mathbb{C} \times !\mathbb{D}$$

$$\begin{aligned} \mathrm{Esp}(\mathbb{B}, \textstyle\bigotimes_{i=1}^{n} \mathbb{C}_i) &\equiv \mathrm{Esp}(\mathbb{B}, \textstyle\sum_{i=1}^{n} \mathbb{C}_i) \\ &= \mathrm{Cat}(!\mathbb{B}, \widehat{\textstyle\sum_{i=1}^{n} \mathbb{C}_i}) \\ &\simeq \mathrm{Cat}(!\mathbb{B}, \textstyle\prod_{i=1}^{n} \hat{\mathbb{C}}_i) \\ &\simeq \prod_{i=1}^{n} \mathrm{Cat}(!\mathbb{B}, \hat{\mathbb{C}}_i) = \textstyle\prod_{i=1}^{n} \mathrm{Esp}(\mathbb{B}, \mathbb{C}_i) \end{aligned}$$

$$\mathrm{Esp}(\mathbb{B}, \mathbb{C} \Rightarrow \mathbb{D}) = \ldots \simeq \ldots = \mathrm{Esp}(\mathbb{B} \otimes \mathbb{C}, \mathbb{D})$$

OBSERVATION 1 : cc-structure of Esp.

def : a cc-bicategory ~~ccc~~ is a bicategory $\mathbb{C}$ equipped with

$$\prod_{i=1}^{n} A_i \quad , \quad A \Rightarrow B$$

for every $A_1 \ldots A_n$, A, B, together with pseudonatural equivalences ~~natural isos~~

$$\mathbb{C}(X, \prod_{i=1}^{n} A_i) \underset{}{\overset{\text{adjoint equivalence}}{\simeq}} \prod_{i=1}^{n} \mathbb{C}(X, A_i)$$

+ biuniversal arrow

$$\mathbb{C}(X \times A, B) \simeq \mathbb{C}(X, A \Rightarrow B)$$

//

eg//

✓ if you have a nice enough cc ! an a smcc $\mathbb{C}$ then $\mathbb{C}_!$. [Paquet]

✓ $\mathrm{Hom}(\mathbb{B}^{op}, \mathrm{Cat})$, games , ...

OBSERVATION 2:

Simply-typed λ-calculus (STLC)

- $x_1 : A_1, \ldots, x_n : A_n \vdash M : B$

program M

$x : \mathbb{N},\ y : \mathbb{N} \vdash x + y : \mathbb{N}$

$$[\![x_1 : A_1, \ldots, x_n : A_n \vdash M : B]\!]$$
$$\shortparallel$$
$$[\![A_1]\!] \times \ldots \times [\![A_n]\!] \xrightarrow{[\![M]\!]} [\![B]\!]$$

typing

$$\frac{\Gamma \longrightarrow \prod_{i=1}^{n} A_i}{(\Gamma \longrightarrow A_i)_{i=1,\ldots,n}}$$

$h \longmapsto \pi_i \circ h$

$\wedge$

$$\frac{\Gamma \longrightarrow (A \Rightarrow B)}{\Gamma \times A \longrightarrow B}$$

evaluation

$$[\![A_1]\!] \times \dots \times [\![A_n]\!] \xrightarrow{\pi_i} [\![A_i]\!]$$

$$\|$$

$$\left[\!\!\left[\ \frac{}{x_1 : A_1 \dots x_n : A_n \vdash x_i : A_i}\ \right]\!\!\right]$$

$(x \longmapsto x+1)$

$\equiv \lambda x.\ x+1$

$$\left[\!\!\left[\ \frac{(\Gamma \vdash M_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \langle M_1, \dots, M_n \rangle : \prod_{i=1}^{n} A_i}\ \right]\!\!\right] = [\![\Gamma]\!] \xrightarrow[\langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle]{} \prod_{i=1}^{n} [\![A_i]\!]$$

$\lambda x.\ M$

$\equiv x \mapsto M$

$$\left[\!\!\left[\ \frac{\Gamma \vdash M : \prod_{i=1}^{n} A_i}{\Gamma \vdash \pi_i(M) : A_i}\ \right]\!\!\right] = [\![\Gamma]\!] \xrightarrow{[\![M]\!]} \prod_{i=1}^{n} [\![A_i]\!] \xrightarrow{\pi_i} [\![A_i]\!]$$

$$\left[\!\!\left[\ \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B}\ \right]\!\!\right] = [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} ([\![A]\!] \Rightarrow [\![B]\!]) \times [\![A]\!] \xrightarrow{\mathrm{eval}} [\![B]\!]$$

$$\left[\!\!\left[\ \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ \right]\!\!\right] = [\![\Gamma]\!] \xrightarrow{\Lambda [\![M]\!]} ([\![A]\!] \Rightarrow [\![B]\!])$$

+ two equations

substitute N for x

$$\underbrace{(\lambda x.\, M)}_{x \mapsto M}\ N =_\beta \overbrace{M[N/x]}\ ,\qquad M =_\eta \lambda x.(M\,x)$$

eg// $(x \mapsto \underbrace{x+1}) 3 = 3+1 = 4$

eg// $f \equiv (x \mapsto f(x))$

+ similar for products

$$\text{typing}\ \frac{\Gamma \longrightarrow \prod_{i=1}^n A_i}{(\Gamma \longrightarrow A_i)_{i=1,\dots,n}} \quad h \mapsto \pi_i \circ h \quad \frac{\Gamma \vdash M : \prod_{i=1}^n A_i}{(\Gamma \vdash N_i : A_i)_{i=1,\dots,n}}$$

$$\lambda\ \frac{\Gamma \longrightarrow (A \Rightarrow B)}{\Gamma \times A \longrightarrow B} \quad \text{evaluation} \quad \frac{\Gamma \vdash M : A \to B}{\Gamma, x : A \vdash N : B}$$

Thm: the free ccc on a set is presented by the syntax of STLC:

obj : types

maps : $x : A \vdash M : B$ modulo $\beta\eta$-eqns.

composition: substitution. //

$\rightsquigarrow$ STLC is sound + complete for reasoning in CCCs.

$$M = N \Rightarrow [\![M]\!] = [\![N]\!]$$

if $[\![M]\!] = [\![N]\!]$ in every model then $M =_{\beta\eta} N$.

eg//

$(X \Rightarrow X)$ is a monoid:

- $\lambda x.\ x$:= identity
- $g \bullet f := \lambda x.\ g(f\ x)$ = composition.

very easy to check this is a monoid. //

Other classic fact:

$$(x \mapsto x+1)\ 3 \overset{\beta}{\Longrightarrow} 3+1$$

directed $\beta+\eta$ laws = running program

Every program will terminate with a normal form.

Can check equality $M \overset{=_{\beta\eta}}{\Longrightarrow} nf(M) = nf(N) \overset{=_{\beta\eta}}{\Longleftarrow} N$ /

Summarising:

1. Esp is a cartesian closed bicategory: it's like a ccc, with equations replaced by (coherent) isomorphisms.

2. (a) cccs have an internal language, STLC, which is sound + complete. This can be a useful way to calculate.

   (b) STLC is normalising: we can run every program P to its normal form, and use this to compute $=_{\beta\eta}$:

$$P =_{\beta\eta} nf(P) \equiv nf(Q) =_{\beta\eta} Q$$

Strategy for proving coherence of Esp:

1. define a version of STLC that's sound + complete for cc-bicategories.

   objects $\sim$ types

   1-cells $\sim$ terms

   2-cells $\sim$ rewrites

2. adapt a proof of normalisation to show this language has a similar property: there's at most 1 rewrite between any two terms.

The language looks like STLC but with $=_{\beta\eta}$ replaced by rewrites $\Gamma \vdash \tau : M \Rightarrow N : B$.

eg// $\Gamma \vdash (\lambda x. M)\, N \underset{\beta}{\Longrightarrow} M\{N/x\} : B$ //

① if there is a rewrite $N \Rightarrow M$ then the corresponding STLC terms are $\beta\eta$-equal

② if $M =_{\beta\eta} N$, then there is a rewrite between the corresponding 2-dimensional terms

<u>lemma</u>: $X \Rightarrow X$ is a pseudomonoid.

<u>Pf</u>: take the proof in a ccc: each $\beta\eta$-eqn corresponds to a 2-cell / rewrite, so we immediately get the req'd isos. By coherence, all the coherence axioms then hold. //

Getting to $\Lambda^{\times,\to}_{ps}$:

① notice that STLC's algebraic structure is described by a clone, with extra structure

↳ replace maps $A \to B$ with multimaps $A_1, \ldots, A_n \to B$

$$\frac{A_1, \ldots, A_n, B \longrightarrow C}{A_1, \ldots, A_n \longrightarrow (B \Rightarrow C)}$$

... for products

⇝ the syntax of STLC is exactly the free such clone

② categorify the definition:

- $A_1, \ldots, A_n \longrightarrow B$ plus
$$\begin{tikzcd} A_1, \ldots, A_n \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd}$$
- associativity and unit laws up to iso

⇝ free such thing is exactly $\Lambda^{\times,\to}_{ps}$

Normalisation: a completely semantic argument.

Idea: construct the right cc-bicategory to talk about terms and their semantic interpretations.

+ See that it has the right structure.

<u>Clone</u> $\mathbb{C}$:

objects $A, B, C, \ldots$

homs $\mathbb{C}(A_1, \ldots, A_n; B)$

substitution $\mathbb{C}(A_1, \ldots, A_n; B) \times \prod_{i=1}^{n} \mathbb{C}(\underline{A}; A_i) \longrightarrow \mathbb{C}(\underline{A}; B)$

projections $p_i : A_1, \ldots, A_n \longrightarrow A_i$

+ 3 axioms.

$\rightsquigarrow$ restrict to length-1 maps you get a category

$\mathrm{Mon}(\mathrm{Esp}[!(\text{set of objects})^{op}, (\text{set of objects})]) \cong$ Clones with that set of objects

$[\mathrm{Bij}, \mathrm{Set}]$

Summary:

- Esp is cartesian closed as a bicat.
- cccs have a nice language

$\rightsquigarrow$ cc-bicats also have a nice language.

$\rightsquigarrow$ this language is coherent: $\exists \leq 1$ rewrite between any two terms

$\rightsquigarrow$ hence: anything you can define in a ccc can be done just as easily in any cc-bicat, e.g. Esp.

Summarising...

- The language $\Lambda_{ps}^{\times,\to}$ is an internal language for cc-bicats.
- It is canonical: derived from a categorification of the algebraic structure of STLC
- It really is "STLC up to iso".

So we know:

- the free cc-bicat. on a signature $= \left\{ \begin{array}{l} \text{obj : types} \\ \text{1-cells : terms} \\ \text{2-cells : rewrites} \end{array} \right\}$ in $\Lambda_{ps}^{\times,\to}$
- hence coherence = "$\exists$ at most one rewrite $\Gamma \vdash \tau : M \Rightarrow N : A$"
  (for any terms M, N)

  a kind of NORMALISATION result!

* cc-bicategories such as Esp are coherent: there is at most one structural iso $\sigma : f \Rightarrow g$ between any two 1-cells.

* constructing objects in a cc-bicategory is no harder than doing so in a ccc

because....

* cc-bicategories have a canonical internal language, which is STLC up to iso ← proof via categorified clones

* the rewrites in this language have normal forms: $\exists$ at most one rewrite $\Gamma \vdash \tau : M \Rightarrow N : A$.

↑ proof by adapting semantic argument for normalising STLC